International Master Programme in Mathematical Physics

Student's guide

2021 – 2022

Master in Mathematical Physics

> D'ailleurs, une science uniquement faite en vue des applications est impossible; les vérités ne sont fécondes que si elles sont enchaînées les unes aux autres. Si l'on s'attache seulement à celles dont on attend un résultat immédiat, les anneaux intermédiaires manqueront, et il n'y aura plus de chaîne.
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> H. Poincaré, *La valeur de la science*

Scientific committee

Vojkan Jakšić (McGill University, Montréal, Canada)
Svetlana Jitomirskaya (University of California, Irvine, USA)
Bruno Nachtergaele (University of California, Davis, USA)
Claude-Alain Pillet (Centre for Theoretical Physics, Marseille, France)
Michael Röckner (Universität Bielefeld, Germany)
Armen Shirikyan (CY Cergy Paris Université, France)
Daniel Ueltschi (University of Warwick, UK)
Simone Warzel (Technische Universität München, Germany)

Directors

Vojkan Jakšić & Armen Shirikyan

Managing directors

Vahagn Nersesyan (University of Versailles, France)
Rafayel Teymurazyan (CMUC, University of Coimbra, Portugal)

GENERAL INFORMATION

Master programme

The master is a university diploma that can be obtained in two years after the bachelor degree. The *Master in Mathematical Physics* is a high-level training whose purpose is to give a solid foundation in various fields of mathematics with an introduction to research. The Master is hosted by the American University of Armenia and is attached to the CY Cergy Paris Université. It operates in collaboration with three partner institutions—Institute of Mathematics of the National Academy of Sciences, Russian–Armenian University, and Yerevan State University.

How it works

The classes take place from the middle of August to the end of June. First-year students have to validate four courses, each of which takes about two months. During the first two weeks, an intensive 15-hour course is given by a visiting lecturer. The lectures are accompanied by exercise sessions, five hours every day, of which two are supervised. The remaining six weeks are devoted to self-study, with weekly one-hour audiovisual contact between the class and the lecturers. A final exam takes place one week after the end of the course.

Second-year students have to validate four courses, including one topic course, and to write a master thesis. The courses are organised in the same way as in the first year. Students have about four months from March to June for writing their thesis.

How to apply

To apply for the master programme, you must have completed at least three years of studies in mathematics after the high school. A four-hour written exam, followed by an interview, takes place before the end of February, and the results are announced by the middle of March. Applicants must register for the exam with Dr. Vahagn Nersesyan before February 19, 2021, by sending their application containing the full academic transcript and a motivation letter.

Students are may choose to enrol with the Russian–Armenian University or Yerevan State University, in which case they may be entitled to a double degree in mathematics. The students not enrolled with one of the two partner universities will be entitled to the diploma of our master programme.

The master is open to all nationalities, and we are committed to the policy of equal opportunities for everyone. Female and disabled candidates are strongly encouraged to apply.

Tuition fee

The tuition conditions for the students already enrolled with one of the partner institutions in the framework of a double degree agreement will be defined by their respective universities. The other applicants need to pay €243 per year as an administrative fee and should contact the directors of the master programme for further inquires.

Scholarships and doctoral studies

All the students who have completed the first year of our master with high grades will be entitled to strong recommendation letters for their applications for a second-year scholarship. Successful students of the master programme will be given an opportunity to do PhD in leading European and North American universities.

Inquiries

All inquires should be addressed to the co-directors of the master programme, Prof. Vojkan Jakšić and Prof. Armen Shirikyan. The most up-to-date information about the master can be found on the web page.

LIST OF COURSES

First year

- Dynamical systems
- Functional analysis
- Partial differential equations
- Probability theory

Second year

- Harmonic analysis
- Spectral theory
- Stochastic analysis

Topic courses

- Information theory and statistical mechanics
- Mathematical problems in turbulence
- Quantum control

Dynamical Systems

LECTURER: Raphaël Krikorian
PhD in Mathematics (École Polytechnique, 1996)

ABSTRACT: The main concern of the theory of Dynamical Systems is to study the evolution of a system (for example a mechanical system) with time. This includes differential equations (the time is continuous), iterations of functions (the time is discrete) and more generally any action of a group on a set. The aim of this course is to describe fundamental concepts that allow for a description of the possible asymptotic behaviors of a dynamical system. We shall thus study various notions of recurrence such as periodicity, transitivity, minimality, ergodicity... We shall also try to define what the term *chaotic behavior* means by introducing the notion(s) of entropy and hyperbolicity. An important framework will be that of Ergodic Theory.

SYLLABUS

1. Topological Dynamical Systems: *Recurrence, transitivity, minimality, topological mixing. Invariant measures, ergodicity, Poincaré recurrence theorem, ergodic theorems (Birkhoff & Von Neumann). Van der Waerden Theorem*
2. Spectral Theory: *Spectral theorem, pure point spectrum, weak mixing*
3. Examples: *In dimension 1: circle homeomorphisms, expanding maps, interval exchange transformations. The role of hyperbolicity. Shifts*
4. Entropy (metric and topological): *Shannon–McMillan–Breiman theorem. K-systems*

REFERENCES

- V. I. ARNOLD, A. AVEZ: *Ergodic Problems of Classical Mechanics*, W. A. Benjamin, 1968.
- P. R. HALMOS: *Lectures on Ergodic Theory*, Chelsea Publishing,1960.
- A. KATOK, B. HASSELBLATT: *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.
- W. PARRY: *Topics in Ergodic Theory*, Cambridge University Press, 2004.
- YA. G. SINAĬ: *Topics in Ergodic Theory*, Princeton University Press, 1994.
- P. WALTERS: *An introduction to Ergodic Theory*, Springer, 1982.

Functional Analysis

LECTURER: Sabine Jansen
PhD in Mathematics (TU Berlin, 2007)

LECTURER: Daniel Ueltschi
PhD in Physics (EPFL, 1998)

ABSTRACT: Many problems in Mathematics lead to linear problems in infinite-dimensional spaces. In this course we shall mainly study infinite-dimensional normed linear spaces and continuous linear transformations between such spaces. We will study Banach spaces and prove the main theorems of this subject (Hahn–Banach, open mapping, uniform boundedness). The last part of the course will be devoted to bounded and unbounded operators, with specific mention of differential operators in L^2 spaces.

SYLLABUS

1. Normed vector spaces: *distance, norm, bases, Lebesgue spaces, linear maps*
2. Linear functionals: *Hahn–Banach theorem, weak convergence, Banach–Alaoglu theorem*
3. Operators in Banach spaces: *Baire Category Theorem, Open Mapping Theorem, Inverse Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Theorem*
4. Hilbert spaces: *Cauchy–Schwarz inequality, polarisation identity, Riesz Representation Theorem, orthonormal basis*
5. Operators in Hilbert spaces: *adjoint, spectrum, compact operators, unbounded operators, Hellinger–Toeplitz Theorem, closed operator, some examples*

REFERENCES

- G. B. FOLLAND: *Real Analysis*, John Wiley, 1999.
- J. K. HUNTER, B. NACHTERGAELE: *Applied Analysis*, World Scientific, 2001.
- E. KREYSZIG: *Introductory Functional Analysis with Applications*, John Wiley, 1989.
- M. REED, B. SIMON: *Functional Analysis (Methods of Modern Mathematical Physics I)*, Elsevier, 1980.
- W. RUDIN: *Functional Analysis*, McGraw-Hill, 1973.

Partial Differential Equations

LECTURER: Vahagn Nersesyan
PhD in Mathematics (University of Paris-Sud, 2008)

ABSTRACT: The aim of this course is to provide a self-contained introduction to partial differential equations (PDEs), mainly focusing on linear equations, but also providing some perspective on nonlinear equations. The course starts with a detailed introduction to the theory of distributions, going from the basic properties of distributions to resolution of some classical PDEs. The second part is devoted to the study of the Fourier transform acting on tempered distributions. In the third part, we apply the tools developed in the first two parts to solve linear versions of some equations of the mathematical physics such as, for example, linear wave, Schrödinger, and heat equations. In the last part, we study the well-posedness of the nonlinear heat equation and some basic properties of its solutions.

SYLLABUS

1. Introduction to distributions: *function spaces, operations on distributions, order, support, first applications to linear PDEs*

2. The Fourier transform: *Schwartz space, tempered distributions, basic properties of the Fourier transform, Paley–Wiener theorem, Sobolev spaces, applications*

3. Linear partial differential equations: *linear wave, Schrödinger, and heat equations, Dirichlet problem for the Laplacian, elements of the spectral theory of the Laplacian*

4. Nonlinear heat equation: *existence and uniqueness of solution of the Cauchy problem, parabolic regularisation, dissipative properties, elements of the theory of attractors*

REFERENCES

- S. ALINHAC, P. GÉRARD: *Pseudo-differential Operators and the Nash-Moser Theorem*, AMS, 2007.
- L. C. EVANS: *Partial Differential Equations*, AMS, 2010.
- G. FOLLAND: *Introduction to Partial Differential Equations*, Princeton, 1995.
- F. G. FRIEDLANDER: *Introduction to the Theory of Distributions*, Cambridge University Press, 1999.
- R. TEMAM: *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer, 1997.

Probability Theory

LECTURER: Armen Shirikyan
PhD in Mathematics (Moscow State University, 1995)

ABSTRACT: We begin with a study of some basic results of probability theory, including Kolmogorov's 0-1 law and limit theorems. The emphasis will be on the ideas, rather than technically complicated results. We next turn to the theory of martingales, for which we establish various inequalities and convergence results. In the final part of the course, we study discrete-time Markov processes. In particular, we deal with the questions of ergodicity and mixing, and describe a general approach due to Doeblin for proving those properties.

SYLLABUS

1. Independence and Kolmogorov's 0-1 law
2. Weak and strong laws of large numbers
3. Cramer's theory of large deviations
4. Convergence of probability measures. Kantorovich–Rubinstein theorem
5. Central limit theorem
6. Conditional expectation. Discrete-time martingales
7. Markov chains
8. Ergodic theory. Doeblin's method of coupling

REFERENCES

- P. BILLINGSLEY: *Probability and Measure*, John Wiley, 1995.
- R. DUDLEY: *Real Analysis and Probability*, Cambridge University Press, 2002.
- T. LINDVALL: *Lectures on the Coupling Method*, Dover, 1992.
- D. STROOCK: *Probability Theory*, Cambridge University Press, 1999.

Harmonic Analysis

LECTURER: Vojkan Jakšić
PhD in Mathematics (Caltech, 1991)

ABSTRACT: The course will focus on topics in classical harmonic analysis that are linked with mathematical physics and in particular spectral theory of self-adjoint operators. Some applications in Fourier analysis will be also discussed.

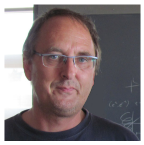
SYLLABUS

1. Poisson transform and Radon–Nikodym derivatives
2. Local L^p norms, $0 < p < 1$
3. Weak convergence
4. Local L^p norms, $p > 1$
5. Local version of the Wiener theorem
6. Poisson representation of harmonic functions
7. The Hardy class $H^\infty(\mathbb{C}_+)$
8. The Borel transforms of measures
9. Spectral theorem—the cyclic case
10. Spectral theory of rank one perturbations

REFERENCES

- H. DYM, H. P. MCKEAN: *Fourier Series and Integrals*, Academic Press, 1972.
- Y. KATZNELSON: *An Introduction to Harmonic Analysis*, Cambridge University Press, 2004.
- V. JAKŠIĆ: *Topics in Spectral Theory*, Open Quantum Systems I. Lecture Notes in Mathematics, vol. 1880, 235–312, Springer, 2006.

Spectral theory



LECTURER: Claude-Alain Pillet
PhD in Physics (ETH-Zürich, 1986)

ABSTRACT: Spectral theory extends to linear operators on infinite dimensional (topological) vector spaces the chapters of linear algebra dealing with eigenvalues and eigenvectors of matrices and their reduction to normal forms (Jordan, triangular, diagonal). This lecture is an introduction to the spectral theory on bounded and unbounded operators on Hilbert spaces. It is strongly biased towards application of this topics to quantum mechanics and quantum field theory.

SYLLABUS

1. Bounded operators on Hilbert spaces and C^*-algebras
2. Compact operators and their spectral decomposition
3. Unbounded operators on Hilbert spaces, Schrödinger operators
4. Functional calculi and the spectral theorem
5. Discrete and essential spectra
6. Perturbation theory of discrete eigenvalues

REFERENCES

- N. DUNFORD, J. T. SCHWARTZ: *Linear Operators II. Spectral Theory*, John Wiley, 1963.
- T. KATO: *Perturbation Theory for Linear Operators*, Springer, 1966.
- M. REED, B. SIMON: *Methods of Modern Mathematical Physics I. Functional Analysis*, Academic Press, 1972.
- M. REED, B. SIMON: *Methods of Modern Mathematical Physics IV. Analysis of Operators*, Academic Press, 1978.
- E. B. DAVIES: *Linear Operators and their Spectra*, Cambridge University Press, 2007.
- B. HELFFER: *Spectral Theory and Applications*, Cambridge University Press, 2013.

Stochastic Analysis

LECTURER: Armen Shirikyan
PhD in Mathematics (Moscow State University, 1995)

ABSTRACT: The aim of this course is to give an introduction to various methods of stochastic analysis. We begin with the general theory of stochastic processes and establish Kolmogorov's continuity theorem, a number of inequalities for martingales, and the Doob–Meyer decomposition for submartingales. We next turn to Ito's integral and stochastic differential equations. We prove the existence and uniqueness of solutions and their strong Markov property. The last part of the course deals with the question of large-time asymptotics for trajectories of Markov processes. We use Doeblin's coupling method to establish the uniqueness of a stationary distribution and its exponential stability as time goes to infinity.

SYLLABUS

1. Generalities on stochastic processes. Kolmogorov continuity theorem
2. Continuous-time martingales. Doob–Meyer decomposition
3. Brownian motion
4. Ito's integral and stochastic differential equations
5. Continuous-time Markov processes and random dynamical systems
6. Diffusion processes: Kolmogorov equation, Girsanov formula.
7. Stationary measures and the problem of mixing

REFERENCES

- I. KARATZAS, S. E. SHREVE: *Brownian Motion and Stochastic Calculus*, Springer, 1991.
- J.-F. LE GALL: *Mouvement Brownien, Martingales et Calcul Stochastique*, Springer, 2013.
- R. KHASMINSKII: *Stochastic Stability of Differential Equations*, Springer, 2012.
- B. ØKSENDAL: *Stochastic Differential Equations*, Springer, 2003.
- A. D. WENTZELL: *A Course in the Theory of Stochastic Processes*, McGraw-Hill, 1981.