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# Large deviations results for the stochastic Navier–Stokes equations

Vahagn Nersesyan \*

## Abstract

In this note, we discuss the problem of large deviations for the stochastic 2D Navier–Stokes equations. We show that the occupation measures of the trajectories of the system satisfy a large deviations principle, provided that the noise acts on all Fourier modes. In the case when the noise is more degenerate and acts on all the determining modes, we obtain an LDP of local type. The proofs use the methods introduced in [13, 20] based on a Kifer-type sufficient condition for LDP and a multiplicative ergodic theorem.

**AMS subject classifications:** 35Q30, 60B12, 60F10

**Keywords:** Navier–Stokes system, white-in-time noise, large deviations principle, occupation measures, coupling

## 0 Introduction

In this note, we review the results of the paper [21], where the large deviations principle (LDP) is studied for the stochastic 2D Navier–Stokes (NS) equations driven by a white-in-time noise

$$\partial_t u + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t, x), \quad \operatorname{div} u = 0, \quad u|_{\partial D} = 0, \quad (0.1)$$

$$u(0, x) = u_0(x), \quad x \in D. \quad (0.2)$$

This system describes the motion of an incompressible fluid in a bounded domain  $D \subset \mathbb{R}^2$  with a smooth boundary  $\partial D$ , where  $\nu > 0$  is the kinematic viscosity,  $u = (u_1(t, x), u_2(t, x))$  and  $p = p(t, x)$  are unknown velocity field and pressure of the fluid,  $f$  is the external (random) force, and  $\langle u, \nabla \rangle = u_1 \partial_1 + u_2 \partial_2$ .

Before stating our results, let us make some comments about this system and recall some previous results on its ergodicity. The system is considered in the usual space of divergence-free vector fields

$$H = \{u \in L^2(D, \mathbb{R}^2) : \operatorname{div} u = 0 \text{ in } D, \langle u, \mathbf{n} \rangle = 0 \text{ on } \partial D\},$$

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where  $\mathbf{n}$  stands for the outward unit normal to  $\partial D$ . By projecting (0.1) to  $H$ , we eliminate the pressure and obtain an evolution equation for the velocity field<sup>1</sup> (e.g., see Section 6 in Chapter 1 of [19])

$$\dot{u} + B(u) + Lu = \Pi f(t, x), \quad (0.3)$$

where  $L = -\Pi\Delta$  is the Stokes operator,  $B(u) = \Pi(\langle u, \nabla \rangle u)$ , and  $\Pi$  is the orthogonal projection onto  $H$  in  $L^2$  (Leray projector).

**Structure of the noise.** We assume that  $f$  is of the form

$$f(t, x) = h(x) + \eta(t, x),$$

where the functions  $h$  and  $\eta$  are, respectively, the deterministic and random components of the external force with  $h \in H$  and

$$\eta(t, x) = \partial_t W(t, x), \quad W(t, x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x). \quad (0.4)$$

Here  $\{b_j\}$  is a sequence in  $\mathbb{R}_+$  such that  $\mathfrak{B}_0 = \sum_{j=1}^{\infty} b_j^2 < \infty$ ,  $\{\beta_j\}$  is a sequence of independent standard Brownian motions defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$  satisfying the usual conditions (see Definition 2.25 in [14]), and  $\{e_j\}$  is an orthonormal basis in  $H$  consisting of the eigenfunctions of  $L$  with eigenvalues  $\{\alpha_j\}$ .

Under these conditions, problem (0.3), (0.2) admits a unique solution and defines a Markov family  $(u_t, \mathbb{P}_u)$  parametrised by the initial condition  $u = u_0 \in H$ . The corresponding Markov semigroups are given by<sup>2</sup>

$$\begin{aligned} \mathfrak{P}_t : C_b(H) &\rightarrow C_b(H), & \mathfrak{P}_t f(u) &= \int_H f(v) P_t(u, dv), \\ \mathfrak{P}_t^* : \mathcal{P}(H) &\rightarrow \mathcal{P}(H), & \mathfrak{P}_t^* \sigma(\Gamma) &= \int_H P_t(v, \Gamma) \sigma(dv), \end{aligned}$$

where  $P_t(u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}$  is the transition function. Recall that a measure  $\mu \in \mathcal{P}(H)$  is *stationary* if  $\mathfrak{P}_t^* \mu = \mu$  for any  $t > 0$ . The existence of a stationary measure is a relatively simple question, it is proved by using the classical Bogolyubov–Krylov argument. The uniqueness and the mixing are much more difficult problems which have been extensively studied in recent years. We refer the reader to the papers [7, 16, 6, 17, 2, 11, 22] and the book [18] for a detailed discussion of this topic. In particular, as it is stated in the following theorem, if  $\eta$  is sufficiently non-degenerate, then the family  $(u_t, \mathbb{P}_u)$  admits a unique stationary measure, which is exponentially mixing (see Theorem 3.5.2 in [18]).

<sup>1</sup>To simplify the notation, we shall assume that  $\nu = 1$ .

<sup>2</sup> $\mathcal{P}(H)$  is the set of probability Borel measures on  $H$  endowed with the topology of the weak convergence.

**Theorem 0.1.** *Under the above conditions, there is an integer  $N = N(\nu, \mathfrak{B}_0, \|h\|)$  such that if*

$$b_j \neq 0 \quad \text{for } j = 1, \dots, N, \quad (0.5)$$

*then the Markov process  $(u_t, \mathbb{P}_u)$  admits a unique stationary measure  $\mu \in \mathcal{P}(H)$ . Moreover,  $\mu$  is exponentially mixing, i.e., there are constants  $C > 0$  and  $\alpha > 0$  such that*

$$\|\mathfrak{P}_t^* \sigma - \mu\|_L^* \leq C e^{-\alpha t} \left( 1 + \int_H \|u\|^2 \sigma(du) \right)$$

*for any  $\sigma \in \mathcal{P}(H)$  and  $t \geq 0$ , where  $\|\cdot\|_L^*$  is the dual-Lipschitz norm.*

This exponential mixing property has many important consequences, such as the strong law of large numbers (SLLN), the law of the iterated logarithm, the central limit theorem, etc. (see Chapter 4 in [18]).

Our goal is to study the large-time behavior of the probabilities of large deviations of trajectories from the stationary measure  $\mu$ . To be more precise, recall that by the SLLN,

$$\mathbb{P}_u \left\{ \frac{1}{t} \int_0^t f(u_s) ds \rightarrow \langle f, \mu \rangle \right\} = 1 \quad \text{as } t \rightarrow \infty$$

for any Hölder-continuous function  $f : H \rightarrow \mathbb{R}$ . This limit implies that for any open set  $O \subset \mathbb{R}$  such that  $\langle f, \mu \rangle \notin \overline{O}$ , we have the following limit

$$\mathbb{P}_u \left\{ \frac{1}{t} \int_0^t f(u_s) ds \in O \right\} \rightarrow 0, \quad t \rightarrow \infty.$$

The large deviations results presented in this note give, in particular, a characterisation of the rate of this convergence: they provide an asymptotic formula of the form

$$\mathbb{P}_u \left\{ \frac{1}{t} \int_0^t f(u_s) ds \in O \right\} = \exp(-ct + o(t)), \quad t \rightarrow \infty$$

with a constant  $c = c(f, O) \geq 0$  that can be expressed in terms of the rate function  $I$  that governs the LDP.

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## 1 Statement of the result

Given a measure  $\nu \in \mathcal{P}(H)$ , we set  $\mathbb{P}_\nu(\Gamma) = \int_H \mathbb{P}_u(\Gamma) \nu(du)$  for any Borel subset  $\Gamma \subset H$  and introduce the following family of *occupation measures*

$$\zeta_t = \frac{1}{t} \int_0^t \delta_{u_s} ds, \quad t > 0$$

defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P}_\nu)$ . Here  $\delta_u$  is the Dirac measure concentrated at  $u \in H$ . We shall say that a mapping  $I : \mathcal{P}(H) \rightarrow [0, +\infty]$  is a *good rate function* if the level set  $\{\sigma \in \mathcal{P}(H) : I(\sigma) \leq \alpha\}$  is compact for any  $\alpha \geq 0$ . For any numbers  $\varkappa > 0$  and  $M > 0$ , we denote

$$\Lambda(\varkappa, M) = \left\{ \nu \in \mathcal{P}(H) : \int_H e^{\varkappa \|v\|^2} \nu(dv) \leq M \right\}.$$

The following theorem is our main result.

**Theorem 1.1.** *Assume that  $\sum_{j=1}^{\infty} \alpha_j b_j^2 < \infty$ ,  $b_j > 0$  for all  $j \geq 1$ , and  $h \in H^1 := H^1(\mathbb{T}^2, \mathbb{R}^2) \cap H$ . Then, there is  $\varkappa > 0$  such that for any  $M > 0$ , the family  $\{\zeta_t, t > 0\}$  satisfies an LDP, uniformly with respect to  $\nu \in \Lambda(\varkappa, M)$ , with a good rate function  $I : \mathcal{P}(H) \rightarrow [0, +\infty]$  not depending on  $\nu$ . More precisely, the following two bounds hold*

**Upper bound.** *For any closed subset  $F \subset \mathcal{P}(H)$ , we have*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \sup_{\nu \in \Lambda} \mathbb{P}_\nu \{\zeta_t \in F\} \leq - \inf_{\sigma \in F} I(\sigma).$$

**Lower bound.** *For any open subset  $G \subset \mathcal{P}(H)$ , we have*

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \inf_{\nu \in \Lambda} \mathbb{P}_\nu \{\zeta_t \in G\} \geq - \inf_{\sigma \in G} I(\sigma).$$

Furthermore,  $I$  is given by

$$I(\sigma) = \sup_{V \in C_b(H)} (\langle V, \sigma \rangle - Q(V)), \quad \sigma \in \mathcal{P}(H),$$

where  $Q : C_b(H) \rightarrow \mathbb{R}$  is a 1-Lipschitz convex function such that  $Q(C) = C$  for any  $C \in \mathbb{R}$ .

This type of large deviations results have been first obtained by Donsker and Varadhan [5] and later extended by many others (see the books [8, 4, 3] and the references therein). The LDP is well understood in the case of finite-dimensional diffusions and Markov processes with compact phase space, provided that the randomness is sufficiently non-degenerate and ensures mixing in the total variation norm. In the context of randomly forced PDE's, the problem of large deviations is mostly considered in the case of vanishing random perturbations and provides estimates for the probabilities of deviations from solutions of the limiting deterministic equations. For the stochastic Burgers and NS equations, the large deviations from a stationary measure is first considered in [9, 10]. In these papers, the force is assumed to be of the form (0.4) with the following condition on the coefficients:

$$cj^{-\alpha} \leq b_j \leq Cj^{-\frac{1}{2}-\varepsilon}, \quad \frac{1}{2} < \alpha < 1, \quad \varepsilon \in \left(0, \alpha - \frac{1}{2}\right]. \quad (1.1)$$

Note that the lower bound in this condition does not allow the coefficients  $b_j$  to converge to zero sufficiently fast. So the external force  $f$  is *irregular* with respect to the space variable, which is not very natural from the physical point of view. The proof is based on a general sufficient condition established in [23], and it essentially uses the *strong Feller property*. The main novelty of Theorem 1.1 is that it proves LDP without a lower bound on the coefficients  $b_j$  as in (1.1) (so we do not have a strong Feller property).

We use the methods introduced in the papers [12, 13], where the LDP is established for a family of dissipative PDE's with parabolic regularisation and a perturbation which is a *regular* random kick force. The proofs of these papers are based on an extension of Kifer's criterion for non-compact spaces and a result on large-time behaviour of generalised Markov semigroups. These results have been later extended in [20] to the case of the stochastic damped nonlinear wave equation (NLW) driven by a spatially regular white noise. The main result of that paper is an LDP of local type.

## 2 Scheme of the proof

Let us briefly present the main ideas of the proof of Theorem 1.1.

*Step 1: Reduction.* The proof is derived from the following three properties.

**Property 1.** For any  $V \in C_b(H)$  and  $\sigma \in \Lambda(\varkappa, M)$ , the following limit exists (called *pressure function*)

$$Q(V) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{E}_\sigma \exp \left( \int_0^t V(u_s) ds \right), \quad (2.1)$$

it does not depend on the initial measure, and is uniform in  $\sigma \in \Lambda(\varkappa, M)$ . Here  $\mathbb{E}_\sigma$  denotes the expectation with respect to  $\mathbb{P}_\sigma$ .

If this property is satisfied, then  $Q : C_b(H) \rightarrow \mathbb{R}$  is a convex 1-Lipschitz function, and its *Legendre transform* is given by

$$I(\sigma) := \begin{cases} \sup_{V \in C_b(H)} (\langle V, \sigma \rangle - Q(V)) & \text{for } \sigma \in \mathcal{P}(H), \\ +\infty & \text{for } \sigma \in \mathcal{M}(H) \setminus \mathcal{P}(H), \end{cases}$$

where  $\mathcal{M}(H)$  is the vector space of signed Borel measures on  $H$  with finite total mass. The function  $I : \mathcal{M}(H) \rightarrow [0, +\infty]$  is convex lower semicontinuous in the weak topology, and  $Q$  can be reconstructed by the formula

$$Q(V) = \sup_{\sigma \in \mathcal{P}(H)} (\langle V, \sigma \rangle - I(\sigma)) \quad \text{for any } V \in C_b(H).$$

We shall say that a measure  $\sigma \in \mathcal{P}(H)$  is an *equilibrium state* for  $V \in C_b(H)$  if it verifies the following equality

$$Q(V) = \langle V, \sigma \rangle - I(\sigma).$$

**Property 2.** There is a vector space  $\mathcal{V} \subset C_b(H)$  such that, for any compact set  $K \subset H$ , the family of restrictions to  $K$  of the functions in  $\mathcal{V}$  is dense in  $C(K)$ , and for any  $V \in \mathcal{V}$  there is a unique equilibrium state  $\sigma_V \in \mathcal{P}(H)$ .

**Property 3.** There is a function  $\Phi : H \rightarrow [0, +\infty]$  with compact level sets  $\{u \in H : \Phi(u) \leq \alpha\}$  for any  $\alpha \geq 0$  such that

$$\mathbb{E}_\sigma \exp\left(\int_0^t \Phi(u_s) ds\right) \leq C e^{ct}, \quad \sigma \in \Lambda(\delta, M), \quad t > 0, \quad (2.2)$$

for some positive constants  $C$  and  $c$ .

Properties 1-3 ensure that the conditions of Kifer's criterion are satisfied, which immediately implies the LDP. Here we use a non-compact version of the criterion, which is established in Theorem 3.3 in [13] (see Theorem 2.1 in [15] for Kifer's original result in the compact case). The main part of the proof of Theorem 1.1 is the verification of these three properties.

*Step 2: Proof of Properties 1-3.* Property 3 is the easiest one. Using some standard a priori estimates for the stochastic NS system, one can see that (2.2) holds with  $\Phi(u) = \varkappa \|u\|_1^2$  if the number  $\varkappa > 0$  is sufficiently small. The function  $\Phi$  has compact level sets, since it is continuous on  $H^1$  and the embedding  $H^1 \subset H$  is compact.

Properties 1 and 2 are derived from a *multiplicative ergodic theorem*. In order to state that result, let us introduce some notation. Let us define the following two *weight functions*

$$\begin{aligned} \mathfrak{w}_m(u) &= 1 + \|u\|^{2m}, \\ \mathfrak{m}_\varkappa(u) &= \exp(\varkappa \|u\|^2), \quad u \in H \end{aligned}$$

for any numbers  $m > 0$  and  $\varkappa > 0$ . To avoid triple subscripts, we shall write  $C_m(H)$  and  $\mathcal{P}_m(H)$  instead of  $C_{m,\varkappa}(H)$  and  $\mathcal{P}_{m,\varkappa}(H)$ . Recall that the *Feynman-Kac semigroup* associated with (0.3), (0.4) is defined by

$$\mathfrak{P}_t^V f(u) = \mathbb{E}_u \left\{ f(u_t) \exp\left(\int_0^t V(u_s) ds\right) \right\}.$$

For sufficiently small  $\varkappa$  and for any  $V \in C_b(H)$ , the application  $\mathfrak{P}_t^V$  maps  $C_m(H)$  into itself. Let  $\mathfrak{P}_t^{V*} : \mathcal{M}_+(H) \rightarrow \mathcal{M}_+(H)$  be its dual. A measure  $\mu \in \mathcal{P}(H)$  is an *eigenvector* if there is  $\lambda > 0$  such that  $\mathfrak{P}_t^{V*} \mu = \lambda^t \mu$  for any  $t > 0$ . Similarly, a function  $h \in C_m(H)$  is an eigenvector if  $\mathfrak{P}_t^V h = \lambda^t h$ . Let  $\mathcal{V}$  be the set of functions  $V \in C_b(H)$  of the form  $V(u) = F(\mathbb{P}_N u)$  for some integer  $N \geq 1$  and a bounded Lipschitz function  $F : H_N \rightarrow \mathbb{R}$ , where  $H_N = \text{span}\{e_1, \dots, e_N\}$  and  $\mathbb{P}_N$  is the orthogonal projection onto  $H_N$  in  $H$ . We have the following result.

**Theorem 2.1** (Multiplicative ergodic theorem). *Under the conditions of Theorem 1.1, for any  $V \in \mathcal{V}$ , there are positive numbers  $m$  and  $\varkappa$ , such that the following assertions hold.*

**Existence and uniqueness.** *The semigroup  $\mathfrak{P}_t^{V*}$  admits a unique eigenvector  $\mu_V \in \mathcal{P}_m(H)$  corresponding to an eigenvalue  $\lambda_V > 0$ . Moreover, the semigroup  $\mathfrak{P}_t^V$  admits a unique positive eigenvector  $h_V \in C_w(H)$  corresponding to  $\lambda_V$  normalised by the condition  $\langle h_V, \mu_V \rangle = 1$ .*

**Convergence.** *For any  $f \in C_m(H)$ ,  $\nu \in \mathcal{P}_w(H)$ , and  $R > 0$ , we have*

$$\lambda_V^{-t} \mathfrak{P}_t^V f \rightarrow \langle f, \mu_V \rangle h_V \quad \text{in } C_b(B_H(R)) \cap L^1(H, \mu_V) \text{ as } t \rightarrow \infty, \quad (2.3)$$

$$\lambda_V^{-t} \mathfrak{P}_t^{V*} \nu \rightarrow \langle h_V, \nu \rangle \mu_V \quad \text{in } \mathcal{M}_+(H) \text{ as } t \rightarrow \infty. \quad (2.4)$$

Moreover, for any  $M > 0$ , the limit

$$\lambda_V^{-t} \mathbb{E}_\nu \left\{ f(u_t) \exp \left( \int_0^t V(u_s) ds \right) \right\} \rightarrow \langle f, \mu_V \rangle \langle h_V, \nu \rangle, \quad t \rightarrow \infty \quad (2.5)$$

holds uniformly in  $\nu \in \Lambda(\varkappa, M)$ .

For any  $V \in \mathcal{V}$ , the existence of limit (2.1) is established by taking  $f = \mathbf{1}$  in (2.3). Then the case of an arbitrary  $V \in C_b(H)$  is obtained by using a simple approximation argument. To show Property 2, we first prove that any equilibrium state  $\sigma_V$  is a stationary measure for the following Markov semigroup:

$$\mathcal{S}_t^V g = \lambda_V^{-t} h_V^{-1} \mathfrak{P}_t^V (gh_V), \quad g \in C_b(H).$$

We then deduce the uniqueness of stationary measure for  $\mathcal{S}_t^V$  from limit (2.4), by showing that  $\sigma_V(dv)$  is given by the formula  $h_V(v) \mu_V(dv)$ .

Thus Theorem 2.1 plays a central role in the proof of the LDP. It is established using an abstract result on large-time asymptotics of generalised Markov semigroups established in Theorem 4.1 in [13] in the discrete-time case and generalised to continuous-time in Theorem 7.4 in [20]. The main ingredients are the following four properties: *uniform irreducibility*, *exponential tightness*, *growth conditions*, and *uniform Feller property*. The verification of the last property is the most technical part of the proof. In the case of the stochastic NS system there are important differences compared to the stochastic damped NLW equations studied in [20]. Here we prove a global LDP, so we need to study the large-time asymptotics of the Feynman–Kac semigroup without any restriction on the smallness of the potential. In particular, this leads to some technical difficulties in the proof of the uniform Feller property. To establish this property, we construct coupling processes using a new auxiliary equation, which allows to have an appropriate Foias–Prodi estimate for the trajectories. Let us also mention that the multiplicative ergodic theorem proved in our case is of slightly more general form and works for a more general class of initial measures.

### 3 Generalisation

In this section, we discuss what happens when the noise does not affect all the Fourier modes, but it is still sufficiently non-degenerate to imply exponential



mixing as in Theorem 0.1. More precisely, we assume that the noise perturbs directly all the determining modes. In this case it is possible to derive an LDP of level 1 type, provided that the system is considered on the torus  $\mathbb{T}^2$ . We have the following result.

**Theorem 3.1.** *Assume that  $\mathfrak{B}_0 = \sum_{j=1}^{\infty} b_j^2 < \infty$  and  $h \in H$ . Then there is an integer  $N = N(\nu, \mathfrak{B}_0, \|h\|) \geq 1$  such that if (0.5) holds, then for any non-constant function  $\psi \in \mathcal{V}$ , there is a number  $\varepsilon = \varepsilon(\psi) > 0$  and a convex function  $I^\psi : \mathbb{R} \rightarrow \mathbb{R}_+$  such that, for any<sup>3</sup>  $u \in H$  and any open subset  $O$  of the interval  $(\langle \psi, \mu \rangle - \varepsilon, \langle \psi, \mu \rangle + \varepsilon)$ , we have*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{P}_u \left\{ \frac{1}{t} \int_0^t \psi(u_s) ds \in O \right\} = - \inf_{\alpha \in O} I^\psi(\alpha). \quad (3.1)$$

Moreover, this limit is uniform with respect to  $u$  in a bounded set of  $H$ .

The proof of this result is obtained by extending the techniques introduced in [20]. According to a local version of the Gärtner–Ellis theorem, limit (3.1) will be established if we show that for some  $\beta_0 > 0$  the following limit exists

$$Q(\beta) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \mathbb{E}_u \exp \left( \int_0^t \beta \psi(u_s) ds \right), \quad |\beta| < \beta_0$$

and it is differentiable in  $\beta$  on  $(-\beta_0, \beta_0)$ . Both properties are derived from a multiplicative ergodic theorem similar to Theorem 2.1. Under condition (0.5), we are able to prove limits (2.3)-(2.5) provided that the potential  $V$  has a sufficiently small oscillation. We work on the torus  $\mathbb{T}^2$  since in the case of a degenerate noise the uniform irreducibility property is deduced from an approximate controllability result established in [1].

The global LDP under condition (0.5) or, more generally, for a highly degenerate noise (i.e., when the integer  $N$  does not depend on the parameters  $\nu, \mathfrak{B}_0$ , and  $\|h\|$ ) is an interesting open problem.

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<sup>3</sup>  $H$  is the space of divergence-free square-integrable vector fields with zero mean on  $\mathbb{T}^2$ .

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