Large deviations principle via Malliavin calculus for the Navier–Stokes system driven by a degenerate white-in-time noise

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January 28, 2022

Abstract

The purpose of this paper is to establish the Donsker–Varadhan type large deviations principle (LDP) for the two-dimensional stochastic Navier–Stokes system. The main novelty is that the noise is assumed to be highly degenerate in the Fourier space. The proof is carried out by using a criterion for the LDP developed in [JNPS18] in a discrete-time setting and extended in [MN18] to the continuous-time. One of the main conditions of that criterion is the uniform Feller property for the Feynman–Kac semigroup, which we verify by using Malliavin calculus.

AMS subject classification: 35Q30, 37H15, 60B12, 60F10, 60H07, 93B05

Keywords: Navier–Stokes system, large deviations, Malliavin calculus, degenerate noise, Feynman–Kac semigroup, approximate controllability, multiplicative ergodicity, uniform Feller property

Contents

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0 Introduction

In this paper, we study the large deviations principle (LDP) for the incompressible Navier–Stokes (NS) system on the torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$\partial_t u - \nu\Delta u + \langle u, \nabla\rangle u + \nabla p = \eta(t,x), \quad \operatorname{div} u = 0, \quad x \in \mathbb{T}^2. \tag{0.1}$$

Here $u = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$ are the unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, and η is an external random force. We consider this system in the usual space

$$H = \left\{ u \in L^2(\mathbb{T}^2, \mathbb{R}^2) : \int_{\mathbb{T}^2} u(x)\mathrm{d}x = 0, \ \operatorname{div} u = 0 \text{ in } \mathbb{T}^2 \right\} \tag{0.2}$$

endowed with the L^2-scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$. Projecting the system (0.1) to the space H, we eliminate the pressure term and obtain the evolution equation

$$\partial_t u - \nu\Delta u + \Pi(\langle u, \nabla\rangle u) = \Pi\eta, \tag{0.3}$$

where Π is the Leray projection to H in $L^2(\mathbb{T}^2, \mathbb{R}^2)$ (see Section 6 in Chapter 1 of [Lio69]). We assume that η is a white-in-time noise of the form

$$\eta(t,x) = \partial_t W(t,x), \quad W(t,x) = \sum_{l \in \mathcal{K}} b_l W_l(t) e_l(x), \tag{0.4}$$

where $\mathcal{K} \subset \mathbb{Z}^2_*$ is a finite set, $\{b_l\}_{l\in\mathcal{K}}$ are non-zero real numbers, $\{W_l\}_{l\in\mathcal{K}}$ are independent standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions (see Definition 2.25 in [KS91]), and

$$e_l(x) = \begin{cases} l^\perp \cos\langle l, x\rangle & \text{if } l_1 > 0 \text{ or } l_1 = 0,\, l_2 > 0, \\ l^\perp \sin\langle l, x\rangle & \text{if } l_1 < 0 \text{ or } l_1 = 0,\, l_2 < 0, \end{cases} \quad l = (l_1, l_2)$$

with $l^\perp = (-l_2, l_1)$. In other words, $\mathcal{K}$ is the collection of the Fourier modes directly perturbed by the noise. Under the above assumptions, the NS system (0.3) defines a family of Markov processes $(u_t, \mathbb{P}_u)$ parametrised by the initial condition $u(0) = u \in H$. The ergodic properties of this family have been extensively studied in the literature. It is now well known that $(u_t, \mathbb{P}_u)$ admits a unique and exponentially mixing stationary measure, provided that the set $\mathcal{K}$ is sufficiently large. Under the condition that $\mathcal{K}$ contains all the determining modes, the ergodicity has been established in different settings in the papers [FM95, KS00, EMS01, KS02, BKL02]. Later, it was shown that the ergodicity remains true for much smaller set $\mathcal{K}$; see the papers [HM06, HM11, FGRT15] for the case when the noise is white-in-time and [KNS20a, KNS20b] for the case of a general bounded noise. The reader is referred to the book [KS12] for more references and for detailed description of different methods.

In this paper, we study the Donsker–Varadhan type LDP for the NS system (0.3). This type of LDP has been extensively studied in the case of finite-dimensional diffusions and Markov processes in compact spaces; see the papers [DV75], the books [FW84, DS89, DZ00], and the references therein. The paper [Wu01] established a general criterion for Donsker–Varadhan type LDP for Markov processes that are strong Feller and irreducible. In that paper the criterion is applied to a class of stochastic damping Hamiltonian systems. There are only few papers considering the problem of LDP for randomly forced PDEs. The first results are obtained in [Gou07a, Gou07b] in the case of the stochastic Burgers and NS equations with strong assumptions on the decay of the coefficients $\{b_l\}$. Indeed, these papers use the criterion of [Wu01], so they require some lower bounds for $\{b_l\}$ in order to guarantee the strong Feller property. These assumptions have been relaxed to the conditions $b_l \neq 0$ for all $l \in \mathbb{Z}^2_*$ and $\sum_{l\in\mathbb{Z}^2_*} |l||b_l|^2 < +\infty$ in the papers [JNPS15, JNPS18], where a family of dissipative PDEs is considered driven by a random kick-force. The proofs of these papers are based on a study of the long-time behaviour of Feynman–Kac semigroup and a Kifer type criterion for the LDP. Under similar non-degeneracy conditions, the local LDP is proved in [MN18] for the stochastic damped non-linear wave equation, and the full LDP is proved in [Ner19] for the stochastic NS system. A controllability approach is used in [JNPS21] to prove the LDP for the Lagrangian trajectories of the NS system. Recently, the criterion of [Wu01] has been used in [WX18] in the case of SPDEs driven by stable type noises and in [WXX21] in the case of non-linear monotone SPDEs with white-in-time noise.

All the papers mentioned above establish the LDP under the assumption that the noise is non-degenerate, i.e., perturbs directly all the Fourier modes in the equation. The goal of the present paper is to establish the LDP in the case of a highly degenerate noise, i.e., when only few Fourier modes are directly perturbed. To formulate our main result, let us recall that a set $\mathcal{K} \subset \mathbb{Z}^2_*$ is a generator if any element of $\mathbb{Z}^2$ is a finite linear combination of elements of $\mathcal{K}$ with integer coefficients. In what follows, we assume that the following condition is satisfied.

(H) *The set $\mathcal{K} \subset \mathbb{Z}^2_*$ in (0.4) is a finite symmetric (i.e., $-\mathcal{K} = \mathcal{K}$) generator that contains at least two non-parallel vectors m and n such that $|m| \neq |n|$.*

This is the condition under which the ergodicity of the NS system is established in [HM06, HM11] in the case of a white-in-time noise and in [KNS20a] in the case of a bounded noise. The set

$$\mathcal{K} = \{(1,0),\, (-1,0),\, (1,1),\, (-1,-1)\} \subset \mathbb{Z}^2_*$$

is an example satisfying this condition.

For any $u \in H$, let us define the family of occupation measures

$$\zeta_t = \frac{1}{t}\int_0^t \delta_{u_s}\mathrm{d}s, \quad t > 0 \tag{0.5}$$

on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_u)$, where δ_v is the Dirac measure concentrated at $v \in H$.

Main Theorem. *Under the Condition (H), the family $\{\zeta_t, t > 0\}$ satisfies the LDP.*

See Theorem 1.1 for more detailed formulation of this result. The proof is carried out by using a criterion for the LDP developed in [JNPS18] in a discrete-time setting and extended in [MN18] to the continuous-time. According to that criterion, the LDP will be established if we show that the following five properties hold for the Feynman–Kac semigroup associated with the NS system (0.3): growth properties, existence of eigenvector, time-continuity, uniform irreducibility, and uniform Feller property. The first three properties are verified in [Ner19] and they hold no matter how degenerate is the noise. The uniform irreducibility property follows from the approximate controllability results obtained in [AS05, AS06]. It is interesting to note that Condition (H) is necessary and sufficient for the approximate controllability of the NS system if one uses controls acting via the Fourier modes in $\mathcal{K}$. The main technical difficulty of this paper is related to the verification of the uniform Feller property, which we carry out by developing the Malliavin calculus analysis of the papers [MP06, HM06, HM11]. More precisely, we derive the uniform Feller property from a gradient estimate for the Feynman–Kac semigroup. The proof of latter contains essential differences with respect to the situations studied in [MP06, HM06, HM11] because of the non-Markovian character of the Feynman–Kac semigroup.

This paper is organised as follows. In Section 1, we explain how the Main Theorem is derived from the above-mentioned five properties. In Section 2, we recall some elements of Malliavin calculus, and in Section 3, we verify the uniform Feller property.

Acknowledgement

Xuhui Peng would like to gratefully thank University of Macau for the hospitality, his research is supported in part by by NNSFC (No. 12071123), Scientific Research Project of Hunan Province Education Department (No. 20A329) and Program of Constructing Key Discipline in Hunan Province. Lihu Xu is supported in part by NNSFC grant (No.12071499), Macao S.A.R grant FDCT 0090/2019/A2 and University of Macau grant MYRG2018-00133-FST.

Notation

In this paper, we use the following notation.

H is the space of divergence-free square-integrable vector fields on $\mathbb{T}^2$ with zero mean value (see (0.2)). It is endowed with the L^2-norm $\|\cdot\|$.

$H^m = H^m(\mathbb{T}^2, \mathbb{R}^2) \cap H$, where $H^m(\mathbb{T}^2, \mathbb{R}^2)$ is the Sobolev space of order $m \geq 1$. We endow the space H^m with the usual Sobolev norm $\|\cdot\|_m$.

$B_{H^m}(a, r)$ is the closed ball in H^m of radius $r > 0$ centred at a. We write $B_{H^m}(r)$ when $a = 0$.

We consider the NS system in the vorticity formulation in the space of square integrable zero mean functions:

$$\tilde{H} = \left\{ w \in L^2(\mathbb{T}^2, \mathbb{R}) : \int_{\mathbb{T}^2} w(x) \mathrm{d}x = 0 \right\} \tag{0.6}$$

equipped with the L^2-norm $\|\cdot\|$. Let $\tilde{H}^m = H^m(\mathbb{T}^2, \mathbb{R}) \cap \tilde{H}$, $m \geq 1$ be endowed with the Sobolev norm denoted by $\|\cdot\|_m$.

$L^\infty(H)$ is the space of bounded Borel-measurable functions $\psi : H \to \mathbb{R}$ with the norm $\|\psi\|_\infty = \sup_{u \in H} |\psi(u)|$. $C_b(H)$ is the space of continuous functions $\psi \in L^\infty(H)$. $C_b^1(H)$ is the space of functions $\psi \in C_b(H)$ that are continuously Fréchet differentiable with bounded derivative.

Let $\mathfrak{w} : H \to [1, +\infty]$ be a Borel-measurable function. Then $C_{\mathfrak{w}}(H)$ (resp., $L^\infty_{\mathfrak{w}}(H)$) is the space of continuous (resp., Borel-measurable) functions $\psi : H \to \mathbb{R}$ such that $\|\psi\|_{L^\infty_{\mathfrak{w}}} = \sup_{u \in H} |\psi(u)|/\mathfrak{w}(u) < +\infty$.

$\mathcal{M}_+(H)$ is the collection of non-negative finite Borel measures on H endowed with the weak convergence topology. For any $\psi \in L^\infty(H)$ and $\mu \in \mathcal{M}_+(H)$, we write $\langle \psi, \mu \rangle = \int_H \psi(u) \mu(\mathrm{d}u)$. $\mathcal{P}(H)$ is the subset of probability measures, and $\mathcal{P}_{\mathfrak{w}}(H)$ is the set of $\mu \in \mathcal{P}(H)$ such that $\langle \mathfrak{w}, \mu \rangle < +\infty$.

$\mathcal{L}(X, Y)$ is the space of linear bounded operators between Banach spaces X and Y endowed with the natural norm $\|\cdot\|_{\mathcal{L}(X,Y)}$.

The letter C is used to denote unessential constants that can change from line to line.

1 Main results

1.1 LDP and multiplicative ergodicity

Recall that a mapping $I : \mathcal{P}(H) \to [0, +\infty]$ is a good rate function if its level sets

$$\{\sigma \in \mathcal{P}(H) : I(\sigma) \leq \alpha\}, \quad \alpha \geq 0$$

are compact. Moreover, if the effective domain of I, defined by

$$D_I = \{\sigma \in \mathcal{P}(H) : I(\sigma) < +\infty\},$$

is not a singleton, we say that I is a non-trivial good rate function. For any $\gamma > 0$ and $M > 0$, we set

$$\Lambda(\gamma, M) := \left\{ \nu \in \mathcal{P}(H) : \int_H e^{\gamma \|u\|^2} \nu(\mathrm{d}u) \leq M \right\}.$$

The following is a more detailed version of the Main Theorem formulated in the Introduction.

Theorem 1.1. *Assume that Condition (H) is verified. Then, for any $\gamma > 0$ and $M > 0$, the family of random probability measures $\{\zeta_t, t > 0\}$ defined by (0.5) satisfies the LDP uniformly w.r.t. the initial measure $\nu \in \Lambda(\gamma, M)$. More precisely, there is a non-trivial good rate function $I : \mathcal{P}(H) \to [0, +\infty]$ that does not depend on γ and M and satisfies the inequalities*

$$\limsup_{t\to+\infty} \frac{1}{t} \log \sup_{\nu\in\Lambda(\gamma,M)} \mathbb{P}_\nu \{\zeta_t \in F\} \leq -\inf_{\sigma\in F} I(\sigma),$$
$$\liminf_{t\to+\infty} \frac{1}{t} \log \inf_{\nu\in\Lambda(\gamma,M)} \mathbb{P}_\nu \{\zeta_t \in G\} \geq -\inf_{\sigma\in G} I(\sigma)$$

for any closed set F and any open set G in $\mathcal{P}(H)$.

This theorem is derived from a multiplicative ergodic theorem for the NS system. To formulate that result, let us introduce the following weight functions:

$$\mathfrak{m}_\gamma(u) = \exp\left(\gamma\|u\|^2\right), \quad \gamma > 0,$$
$$\mathfrak{w}_m(u) = 1 + \|u\|^{2m}, \quad m \geq 1, \ u \in H.$$

There is a constant $\gamma_0 = \gamma_0(\mathfrak{B}_0) > 0$, where $\mathfrak{B}_0 = \sum_{l\in\mathcal{K}} b_l^2$, such that

$$\mathbb{E}_u \mathfrak{m}_\gamma(u_t) \leq e^{-\gamma t}\mathfrak{m}_\gamma(u) + C, \tag{1.1}$$
$$\mathbb{E}_u \mathfrak{w}_m(u_t) \leq e^{-2mt}\mathfrak{w}_m(u) + C \tag{1.2}$$

for any $\gamma \in (0, \gamma_0)$, $m \geq 1$, $u \in H$, and $t \geq 0$, where $C = C(m, \varkappa, \mathfrak{B}_0) > 0$ is a constant; e.g., see Proposition 2.4.9 in [KS12] and Lemma 5.3 in [Ner19] for a proof of these inequalities.

For any $V \in C_b(H)$, the Feynman–Kac semigroup associated with the Markov family $(u_t, \mathbb{P}_u)$ is defined by

$$\mathfrak{P}_t^V \psi(u) = \mathbb{E}_u \left\{ \exp\left(\int_0^t V(u_s)\mathrm{d}s \right) \psi(u_t) \right\}, \quad \mathfrak{P}_t^V : C_b(H) \to C_b(H);$$

its dual is denoted by $\mathfrak{P}_t^{V*} : \mathcal{M}_+(H) \to \mathcal{M}_+(H)$. From (1.1) it follows that $\mathfrak{P}_t^V$ maps the space $C_{\mathfrak{m}_\gamma}(H)$ into itself for $\gamma \in (0, \gamma_0)$.

Theorem 1.2. *Assume that Condition (H) is verified and $V \in C_b^1(H)$. Then there are constants $m = m(V) \geq 1$ and $\gamma = \gamma(\mathfrak{B}_0) \in (0, \gamma_0)$ such that there are unique eigenvectors $h_V \in C_{\mathfrak{w}_m}(H)$ and $\mu_V \in \mathcal{P}_{\mathfrak{m}_\gamma}(H)$ for the semigroups $\mathfrak{P}_t^V$ and $\mathfrak{P}_t^{V*}$ corresponding to an eigenvalue $\lambda_V > 0$, i.e,*

$$\mathfrak{P}_t^{V*}\mu_V = \lambda_V^t \mu_V, \qquad \mathfrak{P}_t^V h_V = \lambda_V^t h_V \qquad \text{for } t > 0,$$

and normalised by $\langle h_V, \mu_V \rangle = 1$. For any $\psi \in C_{\mathfrak{m}_\gamma}(H)$, $\nu \in \mathcal{P}(H)$, and $R > 0$, the following limits hold as $t \to +\infty$:

$$\lambda_V^{-t}\mathfrak{P}_t^V \psi \to \langle \psi, \mu_V \rangle h_V \text{ in } C_b(B_H(R)) \cap L^1(H, \mu_V),$$
$$\lambda_V^{-t}\mathfrak{P}_t^{V*}\nu \to \langle h_V, \nu \rangle \mu_V \text{ in } \mathcal{M}_+(H).$$

Furthermore, for any $M > 0$ and $\varkappa \in (0, \gamma)$,

$$\lambda_V^{-t}\,\mathbb{E}_\nu \left\{ \exp\left(\int_0^t V(u_s)\mathrm{d}s \right) \psi(u_t) \right\} \to \langle \psi, \mu_V \rangle \, \langle h_V, \nu \rangle$$

uniformly w.r.t. $\nu \in \Lambda(\varkappa, M)$ as $t \to +\infty$.

This theorem improves Theorem 1.1 in [Ner19] in two directions. First, in this theorem, the noise is very degenerate, while in [Ner19] all the Fourier modes are assumed to be directly perturbed by the noise. Second, in the present situation, the class of functions V is larger, since the result in [Ner19] applies only to functions depending on finite-dimensional projection of u.

Theorem 1.2 can be viewed as an improvement of Theorem 2.1 in [HM06]. Indeed, in the case $V = 0$, the Feynman–Kac semigroup reduces to the Markov semigroup with eigenvalue $\lambda_V = 1$, eigenvector $h_V = \mathbf{1}$ (the function identically equal to 1 on H), and the measure $\mu_V = \mu$ is the unique stationary measure. The above limits imply that μ is mixing.

Theorem 1.1 is derived from Theorem 1.2 by using a Kifer type criterion in unbounded spaces. Since this derivation is literally the same as in the non-degenerate case (see Section 1 in [Ner19]), we do not give the details. The proof of Theorem 1.2 is discussed in the next subsection.

1.2 Proof of Theorem 1.2

The proof of Theorem 1.2 is carried out by applying a result on large-time asymptotics of generalised Markov semigroups established in [JNPS18] in the discrete-time setting and extended in [MN18] to the continuous-time. Here we apply that result to the Feynman–Kac semigroup $\mathfrak{P}_t^V$ and the associated kernel $P_t^V(u, \Gamma) = (\mathfrak{P}_t^{V*}\delta_u)(\Gamma)$, $u \in H$, $\Gamma \in \mathcal{B}(H)$, where δ_u is the Dirac measure concentrated at u.

By the regularising property of the NS system, the measure $P_t^V(u, \cdot)$ is concentrated on the space H^2 for any $u \in H$ and $t > 0$. For any $R > 0$, let us denote $X_R = B_{H^2}(R)$, and let $V \in C_b(H)$ be arbitrary. Then the following properties hold.

Growth properties. There are numbers $R_0 > 0$, $\gamma \in (0, \gamma_0)$, and $m \geq 1$ such that the following quantities are finite:

$$\sup_{t \geq 0} \frac{\|\mathfrak{P}_t^V \mathfrak{w}_m\|_{L^\infty_{\mathfrak{w}_m}}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}}, \quad \sup_{t \geq 0} \frac{\|\mathfrak{P}_t^V \mathfrak{m}_\gamma\|_{L^\infty_{\mathfrak{m}_\gamma}}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}}, \quad \sup_{t \geq 1} \frac{\|\mathfrak{P}_t^V \Phi\|_{L^\infty_{\mathfrak{m}_\gamma}}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}}, \tag{1.3}$$

where $\|\psi\|_R = \sup_{u \in X_R} |\psi(u)|$ and $\Phi(u) = \|u\|_{H^2}^2$.

Existence of an eigenvector. For any $t > 0$, there is a measure $\mu_{t,V} \in \mathcal{P}(H)$ and a number $\lambda_{t,V} > 0$ such that $\mathfrak{P}_t^{V*}\mu_{t,V} = \lambda_{t,V}\mu_{t,V}$. Moreover, for any numbers $\varkappa \in (0, \gamma_0)$ and $n, m \geq 1$, we have

$$\int_H (\|u\|_{H^2}^n + \mathfrak{m}_\varkappa(u))\, \mu_{t,V}(\mathrm{d}u) < +\infty,$$
$$\|\mathfrak{P}_t^V \mathfrak{w}_m\|_{X_R} \int_{X_R^c} \mathfrak{w}_m(u) \mu_{t,V}(\mathrm{d}u) \to 0 \text{ as } R \to +\infty.$$

Time-continuity. For any $m \geq 1$, $\psi \in C_{\mathfrak{w}_m}(H)$, and $u \in H$, the function $t \mapsto \mathfrak{P}_t^V \psi(u)$, $\mathbb{R}_+ \to \mathbb{R}$ is continuous.

Uniform irreducibility. For any $\rho, r, R > 0$, there are numbers $l = l(\rho, r, R) > 0$ and $p = p(V, \rho, r) > 0$ such that

$$P_l^V(u_0, B_H(\hat{u}, r)) \geq p \tag{1.4}$$

for any $u_0 \in X_R$ and $\hat{u} \in X_\rho$.

Uniform Feller property. The family of functions $\{\|\mathfrak{P}_t^V \mathbf{1}\|_R^{-1} \mathfrak{P}_t^V \psi, t \geq 0\}$ is uniformly equicontinuous [1] on X_R for any $V, \psi \in C_b^1(H)$ and $R \geq R_0$.

The first three of the above properties are established [2] in Propositions 2.1 and 2.5 and Lemma 2.3 in [Ner19]. The proof of the uniform irreducibility is given below, and the uniform Feller property is established in Section 3. Theorem 1.2 is obtained by applying Theorem 7.4 in [MN18] and by literally repeating the arguments of Section 4 in [Ner19].

Proof of uniform irreducibility. Let $P_t(u_0, \cdot)$ be the Markov transition kernel of the family $(u_t, \mathbb{P}_{u_0})$. The boundedness of V implies that

$$P_t^V(u_0, \mathrm{d}v) \geq e^{-t\|V\|_\infty} P_t(u_0, \mathrm{d}v) \quad \text{for } t > 0,\ u_0 \in H. \tag{1.5}$$

According to [AS05, AS06], under Condition (H), the NS system is approximately controllable in the space H by controls taking values in the space

$$\mathcal{H}_\mathcal{K} = \operatorname{span}\{e_l : l \in \mathcal{K}\}.$$

This implies that, for any $u_0, \hat{u} \in H$ and $r > 0$, there is a function $\zeta \in C^\infty([0,1]; \mathcal{H}_\mathcal{K})$ such that

$$\|u(1, u_0, \zeta) - \hat{u}\| < r,$$

where $u(t, u_0, \zeta)$ is the solution of the deterministic NS system (0.3) with the initial condition $u(0) = u_0$ and the (control) force $\eta = \partial_t \zeta$. Using the fact that the mapping $(u_0, \zeta) \mapsto u(1, u_0, \zeta)$ is continuous from $H \times C([0,1]; \mathcal{H}_\mathcal{K})$ to H, the non-degeneracy of the law of the Wiener process W in $C([0,1]; \mathcal{H}_\mathcal{K})$ (i.e., the support of the law of W coincides with the entire space $C([0,1]; \mathcal{H}_\mathcal{K})$), a simple compactness argument, and inequality (1.5), we arrive at (1.4). $\square$

[1] By uniform equicontinuity of $\{\|\mathfrak{P}_t^V \mathbf{1}\|_R^{-1} \mathfrak{P}_t^V \psi, t \geq 0\}$ on X_R we mean that for any $\varepsilon > 0$, there is $\delta > 0$ such that $\|\mathfrak{P}_t^V \mathbf{1}\|_R^{-1} |\mathfrak{P}_t^V \psi(u) - \mathfrak{P}_t^V \psi(u')| < \varepsilon$ for any $u, u' \in X_R$ with $\|u - u'\| < \delta$ and any $t \geq 0$.

[2] These propositions and lemma in [Ner19] are formulated in the case when $X_R = B_{H^1}(R)$ and the noise is non-degenerate. However, their proofs work in the setting of the present paper without any change.

2 Elements of Malliavin calculus

The uniform Feller property is proved by using Malliavin calculus analysis from the papers [MP06, HM06, HM11]. In this section, we recall some basic definitions and estimates from there. To match the framework of these papers, we rewrite the NS system (0.3) in the vorticity formulation:

$$\partial_t w - \nu\Delta w + B(\mathcal{K}w, w) = \sum_{l\in\mathcal{K}} b_l |l|^2 \dot{W}_l(t)\phi_l, \tag{2.1}$$

where $w = \nabla \wedge u$, $B(u, w) = \langle u, \nabla\rangle w$, and $\mathcal{K}$ is the Biot–Savart operator

$$\mathcal{K}w = \sum_{l\in\mathbb{Z}_*^2} |l|^{-2} l^\perp w_{-l}\phi_l$$

with $|l|^2 = l_1^2 + l_2^2$, $l^\perp = (-l_2, l_1)$, $w_l = \langle w, \phi_l\rangle$, and

$$\phi_l(x) = \begin{cases} \sin\langle l, x\rangle & \text{if } l_1 > 0 \text{ or } l_1 = 0,\ l_2 > 0, \\ -\cos\langle l, x\rangle & \text{if } l_1 < 0 \text{ or } l_1 = 0,\ l_2 < 0, \end{cases} \quad l = (l_1, l_2).$$

The operator $\mathcal{K}$ is continuous from $H^s(\mathbb{T}^2; \mathbb{R})$ to $H^{s+1}(\mathbb{T}^2; \mathbb{R}^2)$ for any $s \in \mathbb{R}$; it allows to recover the velocity field from the vorticity via $u = \mathcal{K}w$.

We consider Eq. (2.1) in the space $\tilde{H}$ of real-valued square-integrable functions on $\mathbb{T}^2$ with zero mean value (see (0.6)); it is endowed with the L^2 norm $\|\cdot\|$. Since the underlying probability space plays no role, without loss of generality, we can assume that Ω is the Wiener space, $W(t) = \{W_l(t)\}_{l\in\mathcal{K}}$ is the canonical process, and $\mathbb{P}$ is the Wiener measure. Furthermore, we denote by $\{\theta_l\}_{l\in\mathcal{K}}$ the standard basis in $\mathbb{R}^d$ with $d = |\mathcal{K}|$, and define a linear map $Q : \mathbb{R}^d \to \tilde{H}$ by $Q\theta_l = b_l|l|^2\phi_l$. Let $w_t = \Phi(t, w, W_\cdot)$ be the solution of Eq. (2.1) with initial value $w(0) = w \in \tilde{H}$. For any $0 \le s \le t$ and $\xi \in \tilde{H}$, let $J_{s,t}\xi$ be the solution of the linearised problem:

$$\begin{aligned} \partial_t J_{s,t}\xi - \nu\Delta J_{s,t}\xi + \tilde{B}(w_t, J_{s,t}\xi) &= 0, \\ J_{s,s}\xi &= \xi, \end{aligned} \tag{2.2}$$

where $\tilde{B}(w, v) = B(\mathcal{K}w, v) + B(\mathcal{K}v, w)$.

Recall that, for given $T > 0$ and $v \in L^2([0, T]; \mathbb{R}^d)$, the Malliavin derivative of w_t in the direction v is defined by

$$\mathcal{D}^v w_t = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\Phi(t, w_0, W + \varepsilon\int_0^\cdot v\,\mathrm{d}s) - \Phi(t, w_0, W)\right),$$

where the limit holds almost surely (e.g., see the book [Nua06] for finite-dimensional setting or the papers [MP06, HM06, HM11, FGRT15] for Hilbert space case). By the Riesz representation theorem, there is a linear operator $\mathcal{D} : L^2(\Omega, \tilde{H}) \to L^2(\Omega; L^2([0, T]; \mathbb{R}^d) \otimes \tilde{H})$ such that

$$\mathcal{D}^v w = \langle \mathcal{D}w, v\rangle_{L^2([0,T];\mathbb{R}^d)}. \tag{2.3}$$

On the other hand, we have

$$\mathcal{D}^v w_t = \mathcal{A}_{0,t} v, \tag{2.4}$$

where $\mathcal{A}_{s,t} : L^2([s,t];\mathbb{R}^d) \to \tilde{H}$ is the random operator defined by

$$\mathcal{A}_{s,t} v = \int_s^t J_{r,t} Q v(r) \mathrm{d}r, \quad 0 \le s \le t \le T, \tag{2.5}$$

i.e., $\mathcal{A}_{s,t} v$ is the solution of the linearised problem with a source term:

$$\begin{aligned} \partial_t \mathcal{A}_{s,t} v - \nu \Delta \mathcal{A}_{s,t} v + \tilde{B}(w_t, \mathcal{A}_{s,t} v) &= Q v, \\ \mathcal{A}_{s,s} v &= 0. \end{aligned}$$

The adjoint $\mathcal{A}^*_{s,t} : \tilde{H} \to L^2([s,t];\mathbb{R}^d)$ is given by

$$(\mathcal{A}^*_{s,t} \xi)(r) = Q^* J^*_{r,t} \xi, \quad \xi \in \tilde{H},\, r \in [s,t],$$

where $Q^* : \tilde{H} \to \mathbb{R}^d$ is the adjoint of Q.

Let us denote by $J^{(2)}_{s,t}(\phi,\psi)$ the second derivative of w_t with respect to w in the directions of ϕ and ψ. It is the solution of the problem

$$\begin{aligned} \partial_t J^{(2)}_{s,t}(\phi,\psi) - \nu \Delta J^{(2)}_{s,t}(\phi,\psi) + \tilde{B}(J_{s,t}\phi, J_{s,t}\psi) + \tilde{B}(w_t, J^{(2)}_{s,t}(\phi,\psi)) &= 0, \\ J^{(2)}_{s,s}(\phi,\psi) &= 0. \end{aligned}$$

The next lemma follows from Lemma 4.10 in [HM06].

Lemma 2.1. *For any $\kappa, p > 0$, $0 \le \tau < T$, and $w \in \tilde{H}$, we have*

$$\mathbb{E}_w \sup_{s<t\in[\tau,T]} \|J_{s,t}\|^p_{\mathcal{L}(\tilde{H},\tilde{H})} \le C \exp\{\kappa \|w\|^2\}, \tag{2.6}$$

$$\mathbb{E}_w \sup_{s<t\in[\tau,T]} |||J^{(2)}_{s,t}|||^p \le C \exp\{\kappa \|w\|^2\},$$

where $|||J^{(2)}_{s,t}||| = \sup_{\|\phi\|,\|\psi\|\le 1} \|J^{(2)}_{s,t}(\phi,\psi)\|$, and $C = C(\kappa, p, T-\tau, \mathfrak{B}_0) > 0$ is a constant.

For any $0 \le s < t$, the Malliavin operator is defined by

$$\mathcal{M}_{s,t} = \mathcal{A}_{s,t} \mathcal{A}^*_{s,t} : \tilde{H} \to \tilde{H}.$$

It is a non-negative self-adjoint operator, so its regularisation $\mathcal{M}_{s,t} + \beta \mathbb{I}$ is invertible for any $\beta > 0$. Here $\mathbb{I}$ is the identity. The following lemma gathers some estimates from Section 4.8 in [HM06] and Lemma A.6 in [FGRT15].

Lemma 2.2. *There is a constant $C = C(\mathfrak{B}_0) > 0$ such that, for any $0 \le s < t$, $\beta > 0$, and $w \in \tilde{H}$, we have*

$$\|\mathcal{A}_{s,t}\|^2_{\mathcal{L}(L^2([s,t];\mathbb{R}^d),\tilde{H})} \le C \int_s^t \|J_{r,t}\|^2_{\mathcal{L}(\tilde{H},\tilde{H})} \mathrm{d}r, \tag{2.7}$$

$$\|\mathcal{A}^*_{s,t}(\mathcal{M}_{s,t} + \beta\mathbb{I})^{-1/2}\|_{\mathcal{L}(\tilde{H},L^2([s,t];\mathbb{R}^d))} \le 1, \tag{2.8}$$

$$\|(\mathcal{M}_{s,t} + \beta\mathbb{I})^{-1/2} \mathcal{A}_{s,t}\|_{\mathcal{L}(L^2([s,t];\mathbb{R}^d),\tilde{H})} \le 1, \tag{2.9}$$

$$\|(\mathcal{M}_{s,t} + \beta\mathbb{I})^{-1/2}\|_{\mathcal{L}(\tilde{H},\tilde{H})} \le \beta^{-1/2}. \tag{2.10}$$

We shall use the notation

$$\mathcal{D}_r F = (\mathcal{D}F)(r), \quad \mathcal{D}^j F = (\mathcal{D}F)^j, \quad \mathcal{D}_r^j F = (\mathcal{D}F)^j(r), \quad j = 1, \ldots, d.$$

From the equalities (2.3)-(2.5) it follows that $\mathcal{D}_r^i w_t = J_{r,t} Q\theta_i$, $0 \leq r \leq t$. From this and (2.2), we conclude that, for $0 \leq s < t$,

$$\partial_t \mathcal{D}_r^i J_{s,t}\xi - \nu\Delta\mathcal{D}_r^i J_{s,t}\xi + \tilde{B}(w_t, \mathcal{D}_r^i J_{s,t}\xi) + \tilde{B}(J_{r,t}Q\theta_i, J_{s,t}\xi) = 0.$$

Furthermore, by the variation of constants formula, we have

$$\mathcal{D}_r^i J_{s,t}\xi = \begin{cases} J_{r,t}^{(2)}(Q\theta_i, J_{s,r}\xi) & \text{for } r \geq s, \\ J_{s,t}^{(2)}(J_{r,s}Q\theta_i, \xi) & \text{for } r \leq s. \end{cases}$$

This equality and Lemma 2.1 imply the following lemma. For further details, we refer the reader to Section 4.8 in [HM06] and Lemma A.7 in [FGRT15].

Lemma 2.3. *The operators $J_{s,t}$, $\mathcal{A}_{s,t}$, and $\mathcal{A}_{s,t}^*$ are Malliavin differentiable, and for any $\kappa > 0$, $r \in [s,t]$, $p > 0$, and $w \in \tilde{H}$, the following inequalities hold*

$$\mathbb{E}_w \|\mathcal{D}_r^i J_{s,t}\|_{\mathcal{L}(\tilde{H},\tilde{H})}^p \leq C\exp\{\kappa\|w\|^2\}, \tag{2.11}$$

$$\mathbb{E}_w \|\mathcal{D}_r^i \mathcal{A}_{s,t}\|_{\mathcal{L}(L^2([s,t];\mathbb{R}^d),\tilde{H})}^p \leq C\exp\{\kappa\|w\|^2\}, \tag{2.12}$$

$$\mathbb{E}_w \|\mathcal{D}_r^i \mathcal{A}_{s,t}^*\|_{\mathcal{L}(\tilde{H},L^2([s,t];\mathbb{R}^d))}^p \leq C\exp\{\kappa\|w\|^2\}, \tag{2.13}$$

where $C = C(\kappa, p, t-s, \mathfrak{B}_0) > 0$.

3 Proof of uniform Feller property

3.1 Reduction to a gradient estimate

The aim of this section is to prove the following proposition.

Proposition 3.1. *Under Condition (H), for any $V, \psi \in C_b^1(H)$, there is a number $R_0 = R_0(V) > 0$ such that the family $\left\{\|\mathfrak{P}_t^V \mathbf{1}\|_R^{-1} \mathfrak{P}_t^V \psi, t \geq 0\right\}$ is uniformly equicontinuous on X_R for any $R \geq R_0$.*

Proof. For any $V, \psi \in C_b^1(H)$, let us define functions $\tilde{V}, \tilde{\psi} \in C_b^1(\tilde{H})$ by $\tilde{V}(w) = V(\mathcal{K}w)$ and $\tilde{\psi}(w) = \psi(\mathcal{K}w)$, $w \in \tilde{H}$. The Feynman–Kac semigroup associated with Eq. (2.1) is given by

$$\tilde{\mathfrak{P}}_t^{\tilde{V}} \tilde{\psi}(w) = \mathbb{E}_w \left\{ \exp\left(\int_0^t \tilde{V}(w_s)\mathrm{d}s \right) \tilde{\psi}(w_t) \right\}, \quad \tilde{\mathfrak{P}}_t^{\tilde{V}} : C_b^1(\tilde{H}) \to C_b^1(\tilde{H}).$$

In what follows, the number R_0 is chosen such that the growth properties (1.3) hold. In the next subsection, we prove the following proposition (cf. Proposition 4.3 in [HM06]).

Proposition 3.2. *Under the conditions of Proposition 3.1, for any numbers $\kappa > 0$ and $a \in (0,1)$, there is a constant $C = C(\kappa, a, \|\nabla V\|_\infty, \|V\|_\infty) > 0$ such that*

$$\|\nabla_\xi \tilde{\mathfrak{P}}_t^{\tilde{V}} \tilde{\psi}(w)\| \le C \exp\{\kappa \|w\|^2\} \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \left[\|\nabla\psi\|_\infty a^t + \|\psi\|_\infty\right] \|\xi\| \tag{3.1}$$

for any $w, \xi \in \tilde{H}$ and $t \ge 0$. Here ∇_ξ is the derivative with respect to the initial condition in the direction ξ.

This result implies Proposition 3.1. Indeed, let us take any $u_1, u_2 \in X_R$ and set $w_i = \nabla \wedge u_i$, $i = 1, 2$. Using inequality (3.1) with any $\kappa > 0$ and $a \in (0,1)$ and an interpolation inequality, we see that

$$\begin{aligned}
\left|\mathfrak{P}_t^V \psi(u_1) - \mathfrak{P}_t^V \psi(u_2)\right| &= \left|\tilde{\mathfrak{P}}_t^{\tilde{V}} \tilde{\psi}(w_1) - \tilde{\mathfrak{P}}_t^{\tilde{V}} \tilde{\psi}(w_2)\right| \\
&\le C \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \|w_1 - w_2\| \\
&\le C \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \|u_1 - u_2\|_1 \\
&\le C \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \|u_1 - u_2\|^{1/2} \|u_1 - u_2\|_2^{1/2} \\
&\le C \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \|u_1 - u_2\|^{1/2},
\end{aligned}$$

where $C = C(R, \|V\|_\infty, \|\nabla V\|_\infty, \|\psi\|_\infty, \|\nabla\psi\|_\infty) > 0$. This completes the proof of Proposition 3.1. ☐

3.2 Proof of Proposition 3.2

Let us take any $\xi \in \tilde{H}$ with $\|\xi\| = 1$, denote

$$\Xi_t = \exp\left(\int_0^t \tilde{V}(w_s) \mathrm{d}s\right),$$

and compute the derivative of $\tilde{\mathfrak{P}}_t^{\tilde{V}} \tilde{\psi}(w)$ with respect to w in the direction ξ:

$$\nabla_\xi \tilde{\mathfrak{P}}_t^{\tilde{V}} \tilde{\psi}(w) = \mathbb{E}_w \left[\Xi_t \tilde{\psi}(w_t) \int_0^t \nabla \tilde{V}(w_s) J_{0,s}\xi \mathrm{d}s + \Xi_t \nabla\tilde{\psi}(w_t) J_{0,t}\xi\right]. \tag{3.2}$$

Inspired by the papers [HM06, HM11], the idea of the proof of Proposition 3.2 is to approximate the perturbation $J_{0,t}\xi$ caused by the perturbation ξ of the initial condition with a variation $\mathcal{A}_{0,t}v$ coming from a variation of the noise by an appropriate process v. Let us denote by ρ_t the residual error between $J_{0,t}\xi$ and $\mathcal{A}_{0,t}v$:

$$\rho_t = J_{0,t}\xi - \mathcal{A}_{0,t}v,$$

replace the term $J_{0,t}\xi$ in (3.2) by $\mathcal{A}_{0,t}v + \rho_t$, and recall that $\mathcal{A}_{0,t}v = \mathcal{D}^v w_t$ is the Malliavin derivative of w_t in the direction v. Then, at least formally, using the

Malliavin chain rule (see Proposition 1.2.3 in [Nua06]), we have

$$\begin{aligned}\nabla_\xi \tilde{\mathfrak{P}}_t^{\tilde V}\tilde\psi(w) &= \mathbb{E}_w\left[\Xi_t\tilde\psi(w_t)\int_0^t \nabla\tilde V(w_s)\mathcal{D}^v w_s \mathrm{d}s + \Xi_t\nabla\tilde\psi(w_t)\mathcal{D}^v w_t\right]\\ &\quad + \mathbb{E}_w\left[\Xi_t\tilde\psi(w_t)\int_0^t \nabla\tilde V(w_s)\rho_s \mathrm{d}s + \Xi_t\nabla\tilde\psi(w_t)\rho_t\right]\\ &= \mathbb{E}_w\left[\mathcal{D}^v\left(\Xi_t\tilde\psi(w_t)\right)\right] + \mathbb{E}_w\left[\Xi_t\tilde\psi(w_t)\int_0^t \nabla\tilde V(w_s)\rho_s\mathrm{d}s\right]\\ &\quad + \mathbb{E}_w\left[\Xi_t\nabla\tilde\psi(w_t)\rho_t\right] = I_1+I_2+I_3. \end{aligned}\tag{3.3}$$

The term I_1 is treated using Malliavin integration by parts formula (see Lemma 1.2.1 in [Nua06]):

$$I_1 = \mathbb{E}_w\left[\Xi_t\tilde\psi(w_t)\int_0^t v(s)\mathrm{d}W(s)\right],\tag{3.4}$$

where the stochastic integral $\int_0^t v(s)\mathrm{d}W(s)$ is in the Skorokhod sense. The goal is to choose the process v in a such way that the terms I_i, $i=1,2,3$ are bounded by the right-hand side of inequality (3.1). We use the same choice of v as in the papers [HM06, HM11]. More precisely, for any integer $n\ge 0$, the restriction $v_{n,n+1}$ of the process v to the time interval $[n,n+1]$ is defined by

$$v_{n,n+1}(t) = \begin{cases}\left(\mathcal{A}^*_{n,n+1/2}\left(\mathcal{M}_{n,n+1/2}+\beta\mathbb{I}\right)^{-1}J_{n,n+1/2}\rho_n\right)(t), & t\in[n,n+1/2],\\ 0, & t\in[n+1/2,n+1],\end{cases}\tag{3.5}$$

where we set $\rho_0=\xi$ and $\beta>0$ is a small parameter. This choice allows to have an exponential decay for the moments of ρ_t and of the Skorokhod integral as proved in the following lemmas. Inequality (3.1) is proved by combining these lemmas (with an appropriate choice of parameters therein) and using a growth property of the Feynman–Kac semigroup.

The following two lemmas are versions of Propositions 4.13 and 4.14 in [HM06]. Since their formulations differ from the original ones, we give rather detailed proofs based on the estimates recalled in Section 2.

Lemma 3.3. *For any $\kappa>0$ and $\alpha>0$, there are constants $\beta=\beta(\kappa,\alpha)>0$ and $C=C(\kappa,\alpha)>0$ such that*

$$\mathbb{E}\|\rho_t\|^4 \le C\exp\left\{\kappa\|w\|^2-\alpha t\right\}\quad \textit{for any } w\in\tilde H \textit{ and } t\ge 0.\tag{3.6}$$

Proof. For integer times, this result is established in Proposition 4.13 in [HM06] (this is where Condition (H) is used). Therefore, there are $\beta=\beta(\kappa,\alpha)>0$ and $C=C(\kappa,\alpha)>0$ such that

$$\mathbb{E}\|\rho_n\|^4\le C\exp\{\kappa\|w\|^2-\alpha n\}\quad \text{for any } w\in\tilde H \text{ and } n\ge 0.\tag{3.7}$$

From the construction it follows that

$$\rho_t = \begin{cases} J_{n,t}\rho_n - \mathcal{A}_{n,t}v_{n,t}, & \text{for } t\in[n,n+1/2],\\ J_{n+1/2,t}\rho_{n+1/2}, & \text{for } t\in[n+1/2,n+1]\end{cases}$$

for any $n \geq 0$. Using (3.5) and inequalities (2.8) and (2.10), we get

$$\|v_{n,n+1/2}\|_{L^2([n,n+1/2];\mathbb{R}^d)} \leq \beta^{-1/2}\|J_{n,n+1/2}\rho_n\|. \tag{3.8}$$

Hence, for any $t \in [n, n+1/2]$,

$$\begin{aligned}
\|\rho_t\| &\leq \|J_{n,t}\rho_n\| + \|\mathcal{A}_{n,t}v_{n,t}\| \\
&\leq \|J_{n,t}\rho_n\| + \|\mathcal{A}_{n,t}\|_{\mathcal{L}(L^2([n,t];\mathbb{R}^d),\tilde{H})}\|v_{n,t}\|_{L^2([n,n+1/2];\mathbb{R}^d)} \\
&\leq \|J_{n,t}\rho_n\| + \|\mathcal{A}_{n,t}\|_{\mathcal{L}(L^2([n,t];\mathbb{R}^d),\tilde{H})}\|v_{n,n+1/2}\|_{L^2([n,n+1/2];\mathbb{R}^d)} \\
&\leq C\Big(\|J_{n,t}\rho_n\| + \beta^{-1/2}\|J_{n,n+1/2}\rho_n\| \sup_{s\in[n,t]}\|J_{s,t}\|_{\mathcal{L}(\tilde{H},\tilde{H})}\Big),
\end{aligned} \tag{3.9}$$

where we used (2.7) and (3.8). For any $t \in [n+1/2, n+1]$, it holds that

$$\|\rho_t\| \leq \sup_{s\in[n+1/2,t]}\|J_{s,t}\rho_{n+1/2}\|.$$

Combining this with inequalities (2.6), (3.7), (3.9), the Cauchy–Schwarz inequality, and the fact that $\kappa > 0$ and $\alpha > 0$ are arbitrary, we arrive at (3.6). ☐

Lemma 3.4. *The constants $\beta > 0$ and $C > 0$ in Lemma 3.3 can be chosen such that also*

$$\mathbb{E}\left|\int_n^t v(s)\mathrm{d}W(s)\right|^2 \leq C\exp\{\kappa\|w\|^2 - \alpha n\} \tag{3.10}$$

for any $n \geq 0$, $t \in [n, n+1]$, and $w \in \tilde{H}$.

Proof. In this proof, we consider the endpoint case $t = n+1$; the case $t \in [n, n+1)$ is treated in a similar way. Using the generalised Itô isometry (see Section 1.3 in [Nua06]) and the fact that $v(t) = 0$ for $t \in [n+1/2, n+1]$ (see (3.5)), we obtain

$$\begin{aligned}
\mathbb{E}\left|\int_n^{n+1} v(s)\mathrm{d}W(s)\right|^2 &= \mathbb{E}\int_n^{n+1/2}|v(s)|^2_{\mathbb{R}^d}\mathrm{d}s \\
&\quad + \mathbb{E}\int_n^{n+1/2}\int_n^{n+1/2}\mathrm{Tr}(\mathcal{D}_s v(r)\mathcal{D}_r v(s))\mathrm{d}s\mathrm{d}r \\
&\leq \mathbb{E}\int_n^{n+1/2}|v(s)|^2_{\mathbb{R}^d}\mathrm{d}s \\
&\quad + \mathbb{E}\int_n^{n+1/2}\int_n^{n+1/2}|\mathcal{D}_r v_{n,n+1/2}(s)|^2_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{d}s\mathrm{d}r \\
&= L_1 + L_2.
\end{aligned} \tag{3.11}$$

We estimate L_1 by using (2.6), (3.7), and (3.8):

$$\begin{aligned}
\mathbb{E}\int_n^{n+1/2}|v(s)|^2_{\mathbb{R}^d}\mathrm{d}s &\leq \beta^{-1}\mathbb{E}\|J_{n,n+1/2}\rho_n\|^2 \\
&\leq C\beta^{-1}\exp\{\kappa\|w\|^2/2\}\left(\mathbb{E}\|\rho_n\|^4\right)^{1/2} \\
&\leq C\exp\{\kappa\|w\|^2 - \alpha n/2\}.
\end{aligned} \tag{3.12}$$

To estimate L_2, we use the explicit form of $\mathcal{D}_r v$. Notice that, for any $r \in [n, n+1/2]$ and $i = 1, \ldots, d$,

$$\begin{aligned}
\mathcal{D}_r^i v_{n,n+1/2} = {} & \mathcal{D}_r^i(\mathcal{A}_{n,n+1/2}^*)(\mathcal{M}_{n,n+1/2} + \beta\mathbb{I})^{-1} J_{n,n+1/2}\rho_n \\
& + \mathcal{A}_{n,n+1/2}^*(\mathcal{M}_{n,n+1/2} + \beta\mathbb{I})^{-1} \\
& \quad \times \Big(\mathcal{D}_r^i(\mathcal{A}_{n,n+1/2})\mathcal{A}_{n,n+1/2}^* + \mathcal{A}_{n,n+1/2}\mathcal{D}_r^i(\mathcal{A}_{n,n+1/2}^*)\Big) \\
& \quad \times (\mathcal{M}_{n,n+1/2} + \beta\mathbb{I})^{-1} J_{n,n+1/2}\rho_n \\
& + \mathcal{A}_{n,n+1/2}^*(\mathcal{M}_{n,n+1/2} + \beta\mathbb{I})^{-1}\mathcal{D}_r^i(J_{n,n+1/2})\rho_n.
\end{aligned}$$

By inequalities (2.8)-(2.10), we have

$$\begin{aligned}
\|\mathcal{D}_r^i v_{n,n+1/2}\|_{L^2([n,n+1/2];\mathbb{R}^d)} \leq {} & \beta^{-1}\|\mathcal{D}_r^i(\mathcal{A}_{n,n+1/2})\|_{\mathcal{L}(L^2([n,n+1/2];\mathbb{R}^d),\tilde{H})} \\
& \times \|J_{n,n+1/2}\rho_n\| \\
& + 2\beta^{-1}\|\mathcal{D}_r^i(\mathcal{A}_{n,n+1/2}^*)\|_{\mathcal{L}(\tilde{H},L^2([n,n+1/2];\mathbb{R}^d))} \\
& \times \|J_{n,n+1/2}\rho_n\| \\
& + \beta^{-1/2}\|\mathcal{D}_r^i(J_{n,n+1/2})\rho_n\|.
\end{aligned}$$

Inequalities (2.6), (2.11)-(2.13), and (3.7), imply that

$$\begin{aligned}
\mathbb{E}\int_n^{n+1/2}\int_n^{n+1/2} |\mathcal{D}_r v_{n,n+1/2}(s)|^2_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{d}s\mathrm{d}r & \leq C\beta^{-2}\exp\{\kappa\|w\|^2/2\}\left(\mathbb{E}\|\rho_n\|^4\right)^{1/2} \\
& \leq C\exp\{\kappa\|w\|^2 - n\alpha/2\}. \qquad (3.13)
\end{aligned}$$

Combining estimates (3.11)-(3.13) and using the fact that $\kappa > 0$ and $\alpha > 0$ are arbitrary, we obtain the desired result. $\square$

Finally, we will use a growth estimate for the Feynman–Kac semigroup $\tilde{\mathfrak{P}}_t^{\tilde{V}}$. From the first growth esimate in (1.3) for the semigroup $\mathfrak{P}_t^V$ it follows that there are numbers $R_0 > 0$, $\gamma \in (0, \gamma_0)$, and $m \geq 1$ such that

$$\tilde{\mathfrak{P}}_t^{\tilde{V}}\mathbf{1}(w) \leq C\,\mathfrak{w}_m(\mathcal{K}w)\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0} \quad \text{for any } w \in \tilde{H} \text{ and } t \geq 0. \qquad (3.14)$$

Now we are in a position to prove Proposition 3.2.

Proof of Proposition 3.2. Replacing V by $V - \inf_{v\in H} V(v)$, without loss of generality, we can assume that $V \geq 0$. Let v be the process defined by (3.5), let κ and α be positive numbers (to be chosen later), and let the number $\beta = \beta(\kappa, \alpha) > 0$ be such that inequalities (3.6) and (3.10) hold. Furthermore, let the positive numbers R_0 and m be such that inequality (3.14) holds. Then the computations in (3.3) are rigorously justified, and we need to estimate the terms I_1, I_2, and I_3.

Step 1: Estimate for I_1. We write the Skorokhod integral in the term I_1 (see (3.4)) as follows

$$\int_0^t v(s)\mathrm{d}W(s) = \sum_{n=1}^{\lfloor t\rfloor}\int_{n-1}^n v(s)\mathrm{d}W(s) + \int_{\lfloor t\rfloor}^t v(s)\mathrm{d}W(s),$$

where $\lfloor t \rfloor$ is the largest number less than or equal to t and the sum in the right-hand side is replaced by zero if $t < 1$. Since $v(s)$ is $\mathcal{F}_n$-measurable for $s \in [n-1, n]$, the Skorokhod integral $\int_{n-1}^{n} v(s)\mathrm{d}W(s)$ is also $\mathcal{F}_n$-measurable. Hence, using the Markov property, we obtain

$$\begin{aligned} I_{1,n} &= \mathbb{E}_w\left[\Xi_t\tilde{\psi}(w_t)\int_{n-1}^{n} v(s)\mathrm{d}W(s)\right] \\ &= \mathbb{E}_w\left[\mathbb{E}_w\left(\Xi_t\tilde{\psi}(w_t)\int_{n-1}^{n} v(s)\mathrm{d}W(s)\Big|\mathcal{F}_n\right)\right] \\ &= \mathbb{E}_w\left[\Xi_n\int_{n-1}^{n} v(s)\mathrm{d}W(s)\,\mathbb{E}_w\left(\exp\left\{\int_n^t \tilde{V}(w_s)\mathrm{d}s\right\}\tilde{\psi}(w_t)\Big|\mathcal{F}_n\right)\right] \\ &= \mathbb{E}_w\left[\Xi_n\int_{n-1}^{n} v(s)\mathrm{d}W(s)\,\left(\tilde{\mathfrak{P}}_{t-n}^{\tilde{V}}\tilde{\psi}\right)(w_n)\right] \end{aligned}$$

for any $1 \le n \le \lfloor t \rfloor$. Using inequalities (1.2), (3.10), (3.14), the assumption that $V \ge 0$, and the Cauchy–Schwarz inequality, we see that

$$\begin{aligned} I_{1,n} &\le C\|\psi\|_\infty e^{\|V\|_\infty n}\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0}\mathbb{E}_w\left[\mathfrak{w}_m(u_n)\left|\int_{n-1}^{n} v(s)\mathrm{d}W(s)\right|\right] \\ &\le C\|\psi\|_\infty e^{\|V\|_\infty n}\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0}\left(\mathbb{E}_u\mathfrak{w}_m^2(u_n)\right)^{1/2}\left(\mathbb{E}_w\left|\int_{n-1}^{n} v(s)\mathrm{d}W(s)\right|^2\right)^{1/2} \\ &\le C\|\psi\|_\infty e^{\|V\|_\infty n}\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0}\mathfrak{w}_m(u)\exp\{(\kappa\|w\|^2 - \alpha n)/2\}, \end{aligned}$$

where $u = \mathcal{K}w$ and $u_s = \mathcal{K}w_s$. Next, using (3.10) and $V \ge 0$, we get

$$\begin{aligned} I_{1,\lfloor t \rfloor+1} &= \mathbb{E}_w\left[\Xi_t\tilde{\psi}(w_t)\int_{\lfloor t \rfloor}^{t} v(s)\mathrm{d}W(s)\right] \\ &\le \|\psi\|_\infty e^{\|V\|_\infty t}\mathbb{E}_w\left|\int_{\lfloor t \rfloor}^{t} v(s)\mathrm{d}W(s)\right| \\ &\le C\|\psi\|_\infty e^{\|V\|_\infty t}\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0}\exp\{(\kappa\|w\|^2 - \alpha\lfloor t \rfloor)/2\}. \end{aligned}$$

Combining the estimates for $I_{1,n}$ and $I_{1,\lfloor t \rfloor+1}$, we arrive at

$$I_1 \le C\|\psi\|_\infty\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0}\exp\{\kappa\|w\|^2\}\sum_{n=1}^{\lfloor t \rfloor}\exp\{(\|V\|_\infty - \alpha/2)n\}.$$

Step 2: Estimate for I_2. We first write

$$\begin{aligned} I_2 &= \mathbb{E}_w\left[\int_0^{\lfloor t \rfloor}\Xi_t\tilde{\psi}(w_t)\nabla\tilde{V}(w_s)\rho_s\mathrm{d}s\right] + \mathbb{E}_w\left[\int_{\lfloor t \rfloor}^{t}\Xi_t\tilde{\psi}(w_t)\nabla\tilde{V}(w_s)\rho_s\mathrm{d}s\right] \\ &= I_{2,1} + I_{2,2}. \end{aligned}$$

Let $\lceil s\rceil$ be the smallest integer greater than or equal to s. Then $\rho(s)$ is $\mathcal{F}_{\lceil s\rceil}$-measurable, and using the Markov property, we obtain

$$\begin{aligned} I_{2,1} &= \mathbb{E}_w \left[\int_0^{\lfloor t\rfloor} \mathbb{E}_w \left(\Xi_t \tilde{\psi}(w_t) \nabla \tilde{V}(w_s)\rho_s \middle| \mathcal{F}_{\lceil s\rceil} \right) \mathrm{d}s \right] \\ &= \mathbb{E}_w \left[\int_0^{\lfloor t\rfloor} \Xi_{\lceil s\rceil} \nabla \tilde{V}(w_s)\rho_s \ \mathbb{E}_w \left(\exp\left\{ \int_{\lceil s\rceil}^t \tilde{V}(w_r)\mathrm{d}r \right\} \tilde{\psi}(w_t) \middle| \mathcal{F}_{\lceil s\rceil} \right) \mathrm{d}s \right] \\ &= \mathbb{E}_w \left[\int_0^{\lfloor t\rfloor} \Xi_{\lceil s\rceil} \nabla \tilde{V}(w_s)\rho_s \left(\tilde{\mathfrak{P}}^{\tilde{V}}_{t-\lceil s\rceil} \tilde{\psi} \right)(w_{\lceil s\rceil}) \mathrm{d}s \right]. \end{aligned}$$

Then inequalities (1.2), (3.6), (3.14), the assumption that $V \geq 0$, and the Cauchy–Schwarz inequality imply that

$$\begin{aligned} I_{2,1} &\leq C\|\psi\|_\infty \|\nabla V\|_\infty \|\mathfrak{P}^V_t \mathbf{1}\|_{R_0} \int_0^{\lfloor t\rfloor} e^{\|V\|_\infty \lceil s\rceil} \mathbb{E}_w \left[\mathfrak{w}_m(u_{\lceil s\rceil}) \|\rho_s\| \right] \mathrm{d}s \\ &\leq C\|\psi\|_\infty \|\nabla V\|_\infty \|\mathfrak{P}^V_t \mathbf{1}\|_{R_0} \int_0^{\lfloor t\rfloor} e^{\|V\|_\infty \lceil s\rceil} \left(\mathbb{E}_u \mathfrak{w}^2_m(u_{\lceil s\rceil}) \right)^{1/2} \left(\mathbb{E}_w \|\rho_s\|^2 \right)^{1/2} \mathrm{d}s \\ &\leq C\|\psi\|_\infty \|\nabla V\|_\infty \|\mathfrak{P}^V_t \mathbf{1}\|_{R_0} \mathfrak{w}_m(u) \exp\{\kappa \|w\|^2/4\} \\ &\qquad \times \int_0^{\lfloor t\rfloor} \exp\{\|V\|_\infty \lceil s\rceil - \alpha s/4\} \mathrm{d}s. \end{aligned}$$

To estimate $I_{2,2}$, we use (3.6) and $V \geq 0$:

$$\begin{aligned} I_{2,2} &\leq \|\psi\|_\infty e^{\|V\|_\infty t} \|\nabla V\|_\infty \mathbb{E}_w \int_{\lfloor t\rfloor}^t \|\rho_s\| \mathrm{d}s \\ &\leq C\|\psi\|_\infty e^{\|V\|_\infty t} \|\nabla V\|_\infty \|\mathfrak{P}^V_t \mathbf{1}\|_{R_0} \exp\{(\kappa \|w\|^2 - \alpha t)/4\}. \end{aligned}$$

Thus

$$\begin{aligned} I_2 &\leq C\|\psi\|_\infty \|\nabla V\|_\infty \|\mathfrak{P}^V_t \mathbf{1}\|_{R_0} \exp\{\kappa \|w\|^2\} \\ &\qquad \times \left(\exp\{-\alpha t)/4\} + \int_0^{\lfloor t\rfloor} \exp\{\|V\|_\infty \lceil s\rceil - \alpha s/4\} \mathrm{d}s \right). \end{aligned}$$

Step 3: Estimate for I_3. By (3.6), we have

$$|I_3| \leq C\|\nabla\psi\|_\infty e^{\|V\|_\infty t} \mathbb{E}\|\rho_t\| \leq \|\nabla\psi\|_\infty e^{\|V\|_\infty t} \exp\{(\kappa\|w\|^2 - t\alpha)/4\}.$$

Choosing $\alpha \geq 4\|V\|_\infty - \log a$ and combining the above estimates of the terms I_i, $i = 1, 2, 3$ with (3.3), we complete the proof of Proposition 3.2. □

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