

The complex Ginzburg–Landau equation perturbed by a force localised both in physical and Fourier spaces

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Abstract

In the paper [KNS20a], a criterion for exponential mixing is established for a class of random dynamical systems. In that paper, the criterion is applied to PDEs perturbed by a noise localised in the Fourier space. In the present paper, we show that, in the case of the complex Ginzburg–Landau (CGL) equation, that criterion can be used to consider even more degenerate noise that is localised both in physical and Fourier spaces. This is achieved by checking that the linearised equation is almost surely approximately controllable. We also study the problem of controllability of the nonlinear CGL equation. Using Agrachev–Sarychev type arguments, we prove an approximate controllability property in the case of a control force which is again localised in physical and Fourier spaces.

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0 Introduction

In this paper, we consider the complex Ginzburg–Landau (CGL) equation on the torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$ driven by a very degenerate random or control forces. The main novelty of this paper is the assumption that the force acts on an arbitrary non-empty open set $\omega \subset \mathbb{T}^3$ through only few Fourier modes. More precisely, we consider the problem

$$\partial_t u - (\nu + i)\Delta u + \gamma u + ic|u|^4 u = \chi(x) \eta(t, x), \quad x \in \mathbb{T}^3, \quad (0.1)$$

$$u(0, x) = u_0(x), \quad (0.2)$$

where $\nu, \gamma, c > 0$ are some parameters, $\chi : \mathbb{T}^3 \rightarrow \mathbb{R}_+$ is a smooth function such that $\text{supp } \chi \subset \omega$, and $u = u(t, x)$ is a complex-valued unknown function. Let us begin with the case where η is a random force. To simplify the presentation, we assume in this Introduction that η is a Haar random process of the form

$$\eta(t, x) = \sum_{l \in \mathcal{K}} (b_c^l \eta_c^l(t) \cos\langle l, x \rangle + b_s^l \eta_s^l(t) \sin\langle l, x \rangle), \quad (0.3)$$

where $\mathcal{K} \subset \mathbb{Z}^3$ is the set

$$\mathcal{K} = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}, \quad (0.4)$$

$\{b_c^l, b_s^l\}_{l \in \mathcal{K}}$ are positive numbers, and $\eta_c^l = \eta_{c,1}^l + i\eta_{c,2}^l$ and $\eta_s^l = \eta_{s,1}^l + i\eta_{s,2}^l$ are complex-valued processes with $\{\eta_{c,j}^l, \eta_{s,j}^l : l \in \mathcal{K}, j = 1, 2\}$ being independent copies of a real-valued process given by

$$\tilde{\eta}(t) = \sum_{k=0}^{\infty} \xi_k h_0(t-k) + \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} j^{-q} \xi_{jm} h_{jm}(t).$$

Here $\{\xi_k, \xi_{jm}\}$ are independent identically distributed (i.i.d.) random variables with Lipschitz-continuous density ρ , $\{h_0, h_{jm}\}$ is the Haar basis (see Section 5.2 in [KNS20a]), and $q > 1$. The restriction to integer times of the solution of the problem (0.1), (0.2) defines a family of Markov processes (u_k, \mathbb{P}_u) parametrised by the initial condition $u_0 = u \in H^1(\mathbb{T}^3, \mathbb{C})$. Let $\Lambda \subset \mathbb{T}^3$ be the level set for the maximum of the function χ , i.e.,

$$\Lambda = \{x \in \mathbb{T}^3 : \chi(x) = M\}, \quad \text{where } M = \max_{x \in \mathbb{T}^3} \chi(x). \quad (0.5)$$

We prove the following result.

Theorem A. *Assume that the set Λ has a nonempty interior, the support of the density ρ is bounded, and $\rho(0) > 0$. Then the process (u_k, \mathbb{P}_u) has a unique stationary measure on $H^1(\mathbb{T}^3, \mathbb{C})$ which is exponentially mixing in the dual-Lipschitz metric.*

See Section 1 for more general version of this theorem. The ergodicity of randomly forced PDEs has been extensively studied in the literature, mainly in the case of non-degenerate noises. We refer the reader to the papers [FM95, KS00, EMS01, BKL02] for the first results and the book [KS12] and the reviews [Fla08, Deb13] for further references and discussions of different methods.

We prove Theorem A by using a criterion for ergodicity established in [KNS20a]. According to that result, exponential mixing holds if the resolving operator of the equation has suitable regularity properties, admits one globally stable equilibrium, and has an almost surely non-degenerate derivative (see Conditions (H₁)-(H₃) in Section 1.1). In [KNS20a], that criterion is applied in the case of the Navier–Stokes (NS) system and the CGL equation driven by a noise of the form (0.3) acting on all the domain (i.e., when $\chi \equiv 1$ on \mathbb{T}^3 in the case of Eq. (0.1)). The main difficulty of the problem considered in the present paper is the verification of the non-degeneracy property for the derivative. It is related to the approximate controllability of the linearised equation and is checked combining trigonometric lie-algebraic computations and a unique continuation property for linear parabolic equations. Let us stress that we use in an essential way the local nature of the nonlinear term in the CGL equation, and the problem remains open in the case of the NS system.

The controllability approach used in this paper has been developed starting with the papers [Shi15, Shi21], where exponential mixing is established for the NS system with a space-time or boundary localised noise. In [KNS20b], a version of the criterion of [KNS20a] is derived, where the condition of existence of a globally stable equilibrium is replaced by a weaker property of approximate controllability for the nonlinear equation. The criterion of [KNS20a] is applied in [BGN20] to the system of 3D primitive equations of meteorology and oceanology with a noise only in the temperature equation, and in [Ner19], to the NS system in unbounded domains. In [JNPS19], the controllability approach is further developed to establish a Donsker–Varadhan type large deviations principle for the Lagrangian trajectories of the NS system.

In the case of the Burgers equation driven by a white-in-time noise localised in physical and Fourier spaces, the uniqueness of stationary measure and mixing follow from the approach of [Bor13], although it is not explicitly stated there. For the same equation, but with a forcing that is a sum of an arbitrary smooth deterministic function and a two-dimensional noise localised in any subinterval, the mixing is obtained in [Shi18] using a controllability property to trajectories. The proofs of both papers use in an essential way the strong dissipative character of the Burgers equation and do not work in the case of the CGL equation. Let us also recall that, in the case of the NS system with a white-in-time noise that is degenerate-in-Fourier (but not in physical) space, the Malliavin calculus has been used in the papers [HM06, HM11] to prove exponential mixing for

the NS system. A similar result is obtained in [FGRT15] in the case of the Boussinesq system.

In the second part of this paper, we study the problem of approximate controllability of the nonlinear CGL equation with a control localised both in physical and Fourier spaces. Because of some well-known obstructions, the approximate controllability property does not hold in the entire phase space (e.g., see [DR95, Hen78]). However, using Agrachev–Sarychev type arguments, we show that the restriction of the trajectory to the interior \mathcal{O} of the level set Λ is approximately controllable to any target. More precisely, we prove the following result.

Theorem B. *Let us consider the vector space*

$$\mathcal{H}(\mathcal{K}) = \text{span}\{\cos\langle l, x \rangle, \sin\langle l, x \rangle : l \in \mathcal{K}\}, \quad (0.6)$$

where $\mathcal{K} \subset \mathbb{Z}^3$ is the set given by (0.4). The CGL equation is approximately controllable on the set \mathcal{O} in small time by $\mathcal{H}(\mathcal{K})$ -valued control η , i.e., for any $\varepsilon > 0$, any $T_0 > 0$, and any $u_0, u_1 \in L^2(\mathbb{T}^3, \mathbb{C})$, there is a time $T \in (0, T_0)$, a control $\eta \in L^2([0, T]; \mathcal{H}(\mathcal{K}))$, and a unique solution u of the problem (0.1), (0.2) defined on the interval $[0, T]$ such that

$$\|u(T) - u_0 - u_1 \mathbb{I}_{\mathcal{O}}\|_{L^2} < \varepsilon,$$

where $\mathbb{I}_{\mathcal{O}}$ is the indicator function of the set \mathcal{O} .

In other words, this theorem allows to control approximately the trajectory on \mathcal{O} while keeping it close to the initial condition on $\mathbb{T}^3 \setminus \mathcal{O}$. The local nature of the nonlinearity in the CGL equation is again important for the arguments, and the problem is open in the case of the NS system; see the 7th problem formulated by Agrachev in [Agr14]. In Section 3, we prove different extensions of Theorem B in a more general setting. In particular, we consider the equation in arbitrary space dimension and the degree of the nonlinearity is arbitrary, so the equation is not necessarily globally well-posed. We refer the reader to that section for more details and a short literature review on control problems with finite-dimensional forces.

This paper is organised as follows. In Section 1, we briefly recall the formulation of the abstract criterion of [KNS20a] and apply it to establish exponential mixing for the CGL equation. In Section 2, we verify the non-degeneracy condition by showing that the linearised CGL equation is almost surely approximately controllable. Section 3 is devoted to the study of approximate controllability of the nonlinear CGL equation. Finally, in Section 4, we give examples of saturating spaces for both linear and nonlinear control problems.

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Notation

Let X be a Polish space, that is, a complete separable metric space. We denote by d_X the metric on X and by $B_X(u, R)$ the closed ball of radius $R > 0$ centred at $u \in X$. The Borel σ -algebra on X is denoted by $\mathcal{B}(X)$ and the set of Borel probability measures by $\mathcal{P}(X)$. We use the following spaces, metrics, and norms.

$C_b(X)$ is the space of continuous functions $f : X \rightarrow \mathbb{C}$ endowed with the norm $\|f\|_\infty = \sup_{u \in X} |f(u)|$. We write $C(X)$, when X is compact.

$L_b(X)$ is the space of functions $f \in C_b(X)$ such that

$$\|f\|_{L(X)} = \|f\|_\infty + \sup_{u \neq v} \frac{|f(u) - f(v)|}{d_X(u, v)} < \infty.$$

The dual-Lipschitz metric on $\mathcal{P}(X)$ is defined by

$$\|\mu_1 - \mu_2\|_{L(X)}^* = \sup_{\|f\|_{L(X)} \leq 1} |\langle f, \mu_1 \rangle - \langle f, \mu_2 \rangle|, \quad \mu_1, \mu_2 \in \mathcal{P}(X), \quad (0.7)$$

where $\langle f, \mu \rangle = \int_X f(u) \mu(du)$. Now, assume that X is a Banach space endowed with a norm $\|\cdot\|_X$ and let $J_T = [0, T]$.

$L^p(J_T; X)$, $1 \leq p < \infty$ is the space of measurable functions $f : J_T \rightarrow X$ such that

$$\|f\|_{L^p(J_T; X)} = \left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} < \infty.$$

$L_{\text{loc}}^p(\mathbb{R}_+; X)$, $1 \leq p < \infty$ is the space of measurable functions $f : \mathbb{R}_+ \rightarrow X$ such that $f|_{J_T} \in L^p(J_T; X)$ for any $T > 0$.

$C(J_T; X)$ is the space of continuous functions $f : J_T \rightarrow X$ endowed with the norm

$$\|f\|_{C(J_T; X)} = \sup_{t \in J_T} \|f(t)\|_X.$$

$L^2 = L^2(\mathbb{T}^d; \mathbb{C})$ and $H^s = H^s(\mathbb{T}^d; \mathbb{C})$, $s \geq 0$ are the usual Lebesgue and Sobolev spaces of functions $f : \mathbb{T}^d \rightarrow \mathbb{C}$. We consider L^2 as a real Hilbert space with the scalar product and the norm

$$(u, v)_{L^2} = \text{Re} \int_{\mathbb{T}^d} u(x) \bar{v}(x) dx, \quad \|u\|_{L^2} = \sqrt{(u, u)_{L^2}}$$

and endow the spaces H^s with the corresponding scalar products $(\cdot, \cdot)_{H^s}$ and norms $\|\cdot\|_{H^s}$. Throughout this paper, C denotes unessential positive constants that may change from line to line.

1 Exponential mixing

We start this section by recalling the formulation of the abstract criterion established in [KNS20a]. Then we explain how it is applied to prove exponential mixing for the CGL equation with localised noise.

1.1 Abstract criterion

In this subsection, we consider a random dynamical system of the form

$$u_k = S(u_{k-1}, \eta_k), \quad k \geq 1, \quad (1.1)$$

where $S : H \times E \rightarrow H$ is a continuous mapping, H and E are real separable Hilbert spaces, and $\{\eta_k\}$ are i.i.d. random variables in E . We assume that the law ℓ of η_k has a compact support in E , denoted by \mathcal{K} , and there is a compact set $X \subset H$ such that $S(X \times \mathcal{K}) \subset X$. We consider the Markov process (u_k, \mathbb{P}_u) obtained by restricting the system (1.1) to the set X . The associated Markov operators are denoted by $\mathfrak{P}_k : C(X) \rightarrow C(X)$ and $\mathfrak{P}_k^* : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$. Recall that $\mu \in \mathcal{P}(X)$ is a stationary measure if $\mathfrak{P}_1^* \mu = \mu$. We assume that the following conditions are satisfied for S and $\{\eta_k\}$.

(H₁) There is a Banach space V that is compactly embedded into H such that the mapping $S : H \times E \rightarrow V$ is twice continuously differentiable and its derivatives are bounded on bounded subsets of $H \times E$. Furthermore, the mapping $\eta \mapsto S(u, \eta)$ is analytic from E to H for any fixed $u \in H$, and the derivatives $(D_\eta^j S)(u, \eta)$ are continuous in (u, η) and bounded on bounded subsets of $H \times E$.

(H₂) There is constant $a \in (0, 1)$ and elements $\hat{u} \in X$ and $\hat{\eta} \in \mathcal{K}$ such that

$$\|S(u, \hat{\eta}) - \hat{u}\|_H \leq a \|u - \hat{u}\|_H, \quad u \in X. \quad (1.2)$$

For any $u \in X$, let \mathcal{K}^u be the set of elements $\eta \in \mathcal{K}$ such that the image of the mapping $(D_\eta S)(u, \eta) : E \rightarrow H$ is dense in H . Then $\mathcal{K}^u \in \mathcal{B}(E)$.

(H₃) We have $\ell(\mathcal{K}^u) = 1$ for any $u \in X$.

(H₄) The random variables η_k are of the form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \quad k \geq 1, \quad (1.3)$$

where $b_j > 0$ are such that $\sum_{j=1}^{\infty} b_j^2 < \infty$, $\{e_j\}$ is an orthonormal basis in E , and ξ_{jk} are independent real-valued random variables such that $|\xi_{jk}| \leq 1$ almost surely. Moreover, the law of ξ_{jk} has a Lipschitz-continuous density ρ_j with respect to the Lebesgue measure.

The following is Theorem 1.1 in [KNS20a].

Theorem 1.1. *Under the Conditions (H₁)–(H₄), the process (u_k, \mathbb{P}_u) has a unique stationary measure $\mu \in \mathcal{P}(X)$. Moreover, μ is exponentially mixing in the sense that there are numbers $\sigma > 0$ and $C > 0$ such that*

$$\|\mathfrak{P}_k^* \lambda - \mu\|_{L(X)}^* \leq C e^{-\sigma k}, \quad \lambda \in \mathcal{P}(X), \quad k \geq 1, \quad (1.4)$$

where $\|\cdot\|_{L(X)}^*$ is the dual-Lipschitz metric defined by (0.7).

Usually, in applications the mapping S is the resolving operator of a parabolic PDE (see [KNS20a, KNS20b, Ner19, BGN20]). Conditions (H_1) and (H_2) are standard regularity and dissipativity properties satisfied for a large class of equations. The non-degeneracy Condition (H_3) is less standard; as we will see in Section 2, it can be verified using some control theory arguments.

1.2 Formulation and proof

Let us turn to the CGL equation (0.1). In this section, we assume that η is a random process of the form

$$\eta(t, x) = \sum_{k=1}^{\infty} \mathbb{I}_{[k-1, k)}(t) \eta_k(t - k + 1, x),$$

where $\mathbb{I}_{[k-1, k)}$ is the indicator function of the interval $[k - 1, k)$, and $\{\eta_k\}$ are i.i.d. bounded random variables in $L^2(J; H^2)$ with $J = [0, 1]$. Let

$$S : H^1 \times L^2(J; H^2) \rightarrow H^1, \quad (u_0, \eta_1) \mapsto u(1)$$

be the time-1 resolving operator of the problem (0.1), (0.2), and let (u_k, \mathbb{P}_u) be the Markov process defined by (1.1). Then the following functional

$$\mathcal{H}(u) = \int_{\mathbb{T}^3} \left(\frac{1}{2} |\nabla u(x)|^2 + \frac{c}{6} |u(x)|^6 \right) dx, \quad u \in H^1$$

is a Lyapunov functional for the process (u_k, \mathbb{P}_u) in the sense that there are numbers $c \in (0, 1)$ and $C > 0$ such that

$$\mathbb{E}_u \mathcal{H}(u_k) \leq c^k (1 + \mathcal{H}(u)) + C, \quad u \in H^1, \quad k \geq 1, \quad (1.5)$$

where \mathbb{E}_u is the expectation with respect to \mathbb{P}_u . Inequality (1.5) is obtained from estimate (7.15) in [KNS20a] by taking the expectation. For any $\lambda \in \mathcal{P}(H^1)$, we set

$$\begin{aligned} \mathcal{H}(\lambda) &= \int_{H^1} \mathcal{H}(u) \lambda(du), \\ \mathcal{P}_1(H^1) &= \{\lambda \in \mathcal{P}_1(H^1) : \mathcal{H}(\lambda) < \infty\}. \end{aligned}$$

Let $\Lambda \subset \mathbb{T}^3$ be the level set of the function χ defined by (0.5), and let us assume that the interior \mathcal{O} of Λ is non-empty. We define a notion of \mathcal{O} -saturating subspace as follows. Let $\mathcal{H} \subset H^2$ be a finite-dimensional subspace that is invariant under complex conjugation, i.e., $\bar{\zeta} \in \mathcal{H}$ for all $\zeta \in \mathcal{H}$, and assume that \mathcal{H} contains the function identically equal to 1 on \mathbb{T}^3 . Consider a non-decreasing sequence of finite-dimensional subspaces $\{\mathcal{H}_j\}$ of H^2 defined as follows:

$$\mathcal{H}_0 = \mathcal{H}, \quad \mathcal{H}_j = \text{span}\{\eta, \zeta \xi : \eta, \zeta \in \mathcal{H}_{j-1}, \xi \in \mathcal{H}\}, \quad j \geq 1, \quad (1.6)$$

$$\mathcal{H}_\infty = \cup_{j=0}^{\infty} \mathcal{H}_j. \quad (1.7)$$

Definition 1.2. The subspace \mathcal{H} is said to be \mathcal{O} -saturating if the subspace \mathcal{H}_∞ restricted to the set \mathcal{O} is dense in $L^2(\mathcal{O}; \mathbb{C})$.

Next, we recall the notion of observable function introduced in [KNS20a].

Definition 1.3. Let $\mathcal{H} \subset H^2$ be a finite-dimensional subspace, and let $\{\varphi_l\}_{l \in \mathcal{I}}$ be an orthonormal basis in \mathcal{H} with respect to the scalar product $(\cdot, \cdot)_{H^2}$. A function $\zeta \in L^2(J_T; \mathcal{H})$, where $J_T = [0, T]$, is said to be observable if for any Lipschitz-continuous functions $a_l : J_T \rightarrow \mathbb{R}$, $l \in \mathcal{I}$ and any continuous function $b : J_T \rightarrow \mathbb{R}$, the equality

$$\sum_{l \in \mathcal{I}} a_l(t) (\zeta(t), \varphi_l)_{H^2} - b(t) = 0 \quad \text{in } L^2(J_T; \mathbb{R}) \quad (1.8)$$

implies that $a_l = 0$, $l \in \mathcal{I}$ and $b = 0$ on J_T .

It is easy to see that this definition does not depend on the choice of the basis $\{\varphi_l\}$ in \mathcal{H} .

Theorem 1.4. Let $\chi : \mathbb{T}^3 \rightarrow \mathbb{R}_+$ be a smooth function such that $\mathcal{O} \neq \emptyset$, and let $\mathcal{H} \subset H^2$ be an \mathcal{O} -saturating subspace. Assume that $\{\eta_k\}$ are i.i.d. random variables in $E = L^2(J; \mathcal{H})$ such that the following properties hold.

- The law $\ell \in \mathcal{P}(E)$ of the random variable η_k has a compact support \mathcal{K} in E containing the zero. Moreover, Condition (H₄) is satisfied.
- There is $T \in (0, 1)$ such that the restriction of η_k to the interval J_T is almost surely observable.

Then, for any $\nu, \gamma, c > 0$, the Markov process (u_k, \mathbb{P}_u) has a unique stationary measure $\mu \in \mathcal{P}(H^1)$, and there are numbers $\varkappa > 0$ and $C > 0$ such that

$$\|\mathfrak{P}_k^* \lambda - \mu\|_{L(H^1)}^* \leq C e^{-\varkappa k} (1 + \mathcal{H}(\lambda)), \quad \lambda \in \mathcal{P}_1(H^1), \quad k \geq 1. \quad (1.9)$$

Proof. By Theorem 7.4 in [KNS20a], the process (u_k, \mathbb{P}_u) possesses a compact invariant absorbing set $X \subset H^1$ that is closed and bounded in H^2 . Conditions (H₁)–(H₄) in Theorem 1.1 are satisfied for the restriction of (u_k, \mathbb{P}_u) to X if we take $H = H^1$ and $E = L^2(J; \mathcal{H})$. Indeed, the verification of Conditions (H₁) and (H₂) is carried out in the same way as in the case $\chi \equiv 1$ considered in Theorem 4.7 in [KNS20a], Condition (H₃) is verified in the next section, and (H₄) holds by assumption. Thus, by Theorem 1.1, we have exponential mixing (1.4). Then the validity of (1.9) follows from (1.5) and the regularisation property of the CGL equation; e.g., see Section 3.2.4 in [KS12] for a similar argument in the case of the NS system. \square

For any finite set $\mathcal{I} \subset \mathbb{Z}^3$ containing the zero vector, let us consider the subspace

$$\mathcal{H}(\mathcal{I}) = \text{span}\{\cos\langle l, x \rangle, \sin\langle l, x \rangle : l \in \mathcal{I}\}. \quad (1.10)$$

Let $\tilde{\mathcal{I}}$ be the set of all linear combinations of vectors in \mathcal{I} with integer coefficients. Recall that \mathcal{I} is a generator if $\tilde{\mathcal{I}} = \mathbb{Z}^3$. By Proposition 4.1, the subspace $\mathcal{H}(\mathcal{I})$ is

\mathcal{O} -saturating if and only if \mathcal{I} is a generator. In particular, $\mathcal{H}(\mathcal{K})$ is \mathcal{O} -saturating, where $\mathcal{K} \subset \mathbb{Z}^3$ is defined by (0.4). Furthermore, in Section 5.2 in [KNS20a], it is proved that the Haar process (0.3) satisfies the observability condition. Thus, Theorem A formulated in the Introduction follows as a consequence of Theorem 1.4.

2 Controllability of the linearised equation

We use the same notation as in the previous section. The objective of this section is to prove the following result.

Theorem 2.1. *Under the conditions of Theorem 1.4, for any $u \in X$ and ℓ -almost every $\eta \in E$, the image of the linear mapping $(D_\eta S)(u, \eta) : E \rightarrow H^1$ is dense in H^1 .*

Proof. Let us denote by $\tilde{u} \in W^{1,2}(J; H^1) \cap L^2(J; H^3)$ the solution of Eq. (0.1) corresponding to the initial condition $\tilde{u}(0) = u \in X$ and consider the linearised problem

$$\dot{v} + Lv + Q(\tilde{u}; v) = \chi g, \quad v(0) = 0, \quad (2.1)$$

where

$$\begin{aligned} L &= -(\nu + i)\Delta + \gamma, \\ Q(u; v) &= ic(3|u|^4v + 2|u|^2u^2\bar{v}). \end{aligned} \quad (2.2)$$

For any $g \in E$, let¹ $A_t g$ be the solution of the problem (2.1). The theorem will be proved if we show that the vector space $\{A_1 g : g \in E\}$ is dense in H^1 for ℓ -a.e. $\eta \in E$. Let us fix a realisation of η and a time $T \in (0, 1)$ such that the observability property holds on the interval J_T . We are going to show that the space $\{A_T g : g \in E\}$ is dense in L^2 . This, combined with the parabolic regularisation and a density property of the set of solutions of linear parabolic equations proved in Proposition 7.2 in [KNS20a], will imply the required property.

Let $R(t, s) : L^2 \rightarrow L^2$, $0 \leq s \leq t \leq T$ be the resolving operator of the homogeneous problem

$$\dot{v} + Lv + Q(\tilde{u}; v) = 0, \quad v(s) = v_0.$$

Then, by the Duhamel formula, we have

$$A_t g = \int_0^t R(t, s) \chi g(s) ds.$$

Moreover, the function

$$w(s) = R^*(T, s)w_0, \quad (2.3)$$

¹To simplify the notation, we do not indicate the dependence of A_t and other quantities on \tilde{u} .

where $R(t, s)^* : L^2 \rightarrow L^2$ is the adjoint of the operator $R(t, s)$ in L^2 , is the solution of the backward problem

$$\dot{w} - L^*w - Q^*(\tilde{u}; w) = 0, \quad w(T) = w_0 \quad (2.4)$$

with $L^* = -(\nu - i)\Delta + \gamma$ and

$$Q^*(\tilde{u}; w) = ic(-3|u|^4w + 2|u|^2u^2\bar{w}).$$

Let $\mathbb{P}_{\mathcal{H}} : L^2 \rightarrow L^2$ be the orthogonal projection onto \mathcal{H} in L^2 . We need to prove that the image of the linear operator

$$\mathcal{A} : L^2(J_T; L^2) \rightarrow L^2, \quad \mathcal{A} = A_T \mathbb{P}_{\mathcal{H}}$$

is dense in L^2 . It suffices to show that the kernel of the adjoint operator

$$\mathcal{A}^* : L^2 \rightarrow L^2(J_T; L^2), \quad \mathcal{A}^* = \mathbb{P}_{\mathcal{H}} \chi R(T, s)^*$$

is trivial. To prove this, let us take any $w_0 \in \text{Ker } \mathcal{A}^*$. Then $\mathbb{P}_{\mathcal{H}} \chi R(T, t)^* w_0 = 0$ for any $t \in J_T$, hence (cf. (2.3))

$$(\chi \zeta, w(t))_{L^2} = 0, \quad t \in J_T$$

for any $\zeta \in \mathcal{H}_0 = \mathcal{H}$. Now, assuming that we have the relation

$$(\chi^p \zeta, w(t))_{L^2} = 0, \quad t \in J_T \quad (2.5)$$

for any $\zeta \in \mathcal{H}_k$ and some $p \geq 1$, let us prove that

$$(\chi^{p+4} \xi, w(t))_{L^2} = 0, \quad t \in J_T \quad (2.6)$$

for any $\xi \in \mathcal{H}_{k+1}$. Indeed, differentiating (2.5) in t and using Eq. (2.4), we obtain

$$(L(\chi^p \zeta) + \chi^p Q(\tilde{u}(t); \zeta), w(t))_{L^2} = 0, \quad t \in J_T. \quad (2.7)$$

Let us put $\eta^l(t) = (\eta(t), \varphi_l)_{H^2}$, $l \in \mathcal{I}$ and write

$$\eta(t) = \sum_{l \in \mathcal{I}} \eta^l(t) \varphi_l.$$

Differentiating (2.7) in t and using Eqs. (0.1) and (2.4), we get

$$\begin{aligned} & (L(\chi^p \zeta) + \chi^p Q(\tilde{u}; \zeta), \dot{w})_{L^2} - (\chi^p B_2(\tilde{u}; \zeta, L\tilde{u} + B(\tilde{u})), w)_{L^2} \\ & + \sum_{l \in \mathcal{I}} (\chi^{p+1} B_2(\tilde{u}; \zeta, \varphi_l), w)_{L^2} \eta^l(t) = 0, \end{aligned}$$

where $B_k(u; \cdot)$ is the k^{th} derivative of $B(u) = ic|u|^4u$ (so $B_1(u; \cdot) = Q(u; \cdot)$ and $B_5(\tilde{u}; \cdot) = B_5(\cdot)$ is independent of \tilde{u}). Thus, we have (1.8), where

$$\begin{aligned} a_l(t) &= (\chi^{p+1} B_2(\tilde{u}(t); \zeta, \varphi_l), w(t))_{L^2}, \\ b(t) &= (L(\chi^p \zeta) + \chi^p Q(\tilde{u}(t); \zeta), \dot{w}(t))_{L^2} \\ &\quad - (\chi^p B_2(\tilde{u}(t); \zeta, L\tilde{u}(t) + B(\tilde{u}(t))), w(t))_{L^2}. \end{aligned}$$

Then the functions $a_l, l \in \mathcal{I}$ are Lipschitz-continuous and b is continuous, so the observability assumption implies that

$$(\chi^{p+1} B_2(\tilde{u}(t); \zeta, \varphi_l), w(t))_{L^2} = 0, \quad l \in \mathcal{I}, t \in J_T.$$

Iterating the same argument three more times, we get

$$(\chi^{p+4} B_5(\zeta, \varphi_l, \varphi_j, \varphi_m, \varphi_n), w(t))_{L^2} = 0, \quad j, l, m, n \in \mathcal{I}, t \in J_T.$$

Using the equality

$$B_5(\zeta, \xi, 1, 1, 1) = 12ic(3\zeta\xi + \bar{\zeta}\bar{\xi} + 3\bar{\zeta}\xi + 3\zeta\bar{\xi})$$

and the facts that $1 \in \mathcal{H}$ and the spaces \mathcal{H} and \mathcal{H}_k are invariant under complex conjugation, we arrive at (2.6). Thus, we proved the following property: for any $\zeta \in \mathcal{H}_\infty$, there is a sequence of integers $p_n \rightarrow +\infty$ as $n \rightarrow +\infty$ such that

$$(\chi^{p_n} \zeta, w(t))_{L^2} = 0, \quad n \geq 1, t \in J_T.$$

Dividing this equality by M^{p_n} , where $M = \max_{x \in \mathbb{T}^3} \chi(x)$, passing to the limit as $n \rightarrow +\infty$, and using the Lebesgue theorem on dominated convergence, we obtain

$$(\zeta, w(t))_{L^2(\mathcal{O}; \mathbb{C})} = 0, \quad t \in J_T.$$

From the saturation property it follows that $w(t, x) = 0$ for any $t \in J_T$ and $x \in \mathcal{O}$. The unique continuation property for parabolic equations (e.g., see [SS87]) implies that $w(t, x) = 0$ for any $t \in J_T$ and $x \in \mathbb{T}^3$. In particular, $w(T) = w_0 = 0$ (see (2.4)). Thus $\text{Ker } \mathcal{A}^* = \{0\}$, which completes the proof of the theorem. \square

3 Controllability of the nonlinear equation

3.1 Preliminaries

In this section, we consider the problem of controllability of the following nonlinear CGL equation on the torus of arbitrary dimension $d \geq 1$:

$$\partial_t u + Lu + B(u) = f(t, x), \quad x \in \mathbb{T}^d, \quad (3.1)$$

where $\nu > 0$ and $\gamma \geq 0$ are some parameters, L is defined by (2.2), and $B(u)$ denotes the nonlinear term $ic|u|^{2p}u$ with arbitrary integer $p \geq 1$ and parameter $c > 0$. This equation is supplemented with the initial condition

$$u(0, x) = u_0(x) \quad (3.2)$$

which is assumed to belong to a Sobolev space H^s of integer order $s > d/2$, so that the Cauchy problem is locally well-posed in the sense of the following proposition. For any $T > 0$, let us introduce the space

$$\mathcal{X}_T = C(J_T; H^s) \cap L^2(J_T; H^{s+1})$$

²In this section, we do not assume that the unforced equation admits one globally stable equilibrium, so the value $\gamma = 0$ is allowed.

endowed with the norm

$$\|u\|_{\mathcal{X}_T} = \|u\|_{C(J_T; H^s)} + \|u\|_{L^2(J_T; H^{s+1})}.$$

Together with Eq. (3.1), we consider the following more general equation:

$$\partial_t u + L(u + \zeta) + B(u + \zeta) = f(t, x). \quad (3.3)$$

Proposition 3.1. *For any $\hat{u}_0 \in H^s$, $\hat{\zeta} \in C(\mathbb{R}_+; H^{s+1})$, and $\hat{f} \in L^2_{\text{loc}}(\mathbb{R}_+; H^{s-1})$, there is a time $T_* := T_*(\hat{u}_0, \hat{\zeta}, \hat{f}) > 0$ and a unique solution \hat{u} of the problem (3.3), (3.2) with data $(u_0, \zeta, f) = (\hat{u}_0, \hat{\zeta}, \hat{f})$ whose restriction to the interval J_T belongs to the space \mathcal{X}_T for any $T < T_*$. Furthermore, there are constants $\delta = \delta(T, \lambda) > 0$ and $C = C(T, \lambda) > 0$, where*

$$\lambda = \|\hat{\zeta}\|_{C(J_T; H^{s+1})} + \|\hat{f}\|_{L^2(J_T; H^{s-1})} + \|\hat{u}\|_{\mathcal{X}_T},$$

such that

- for any $u_0 \in H^s$, $\zeta \in C(J_T; H^{s+1})$, and $f \in L^2(J_T; H^{s-1})$ satisfying

$$\|u_0 - \hat{u}_0\|_s + \|\zeta - \hat{\zeta}\|_{C(J_T; H^{s+1})} + \|f - \hat{f}\|_{L^2(J_T; H^{s-1})} < \delta, \quad (3.4)$$

the problem (3.3), (3.2) has a unique solution $u \in \mathcal{X}_T$;

- let \mathcal{R} be the mapping taking (u_0, ζ, f) satisfying (3.4) to the solution u . Then

$$\|\mathcal{R}(u_0, \zeta, f) - \mathcal{R}(\hat{u}_0, \hat{\zeta}, \hat{f})\|_{\mathcal{X}_T} \leq C (\|u_0 - \hat{u}_0\|_s + \|\zeta - \hat{\zeta}\|_{C(J_T; H^{s+1})} + \|f - \hat{f}\|_{L^2(J_T; H^{s-1})}).$$

This proposition is proved by literally repeating the arguments of the proof of Proposition 1 in [Ner21b], where parabolic equation is considered with a real-valued polynomially growing nonlinearity. In what follows, we assume that the source term is of the form $f = h + \chi\eta$:

$$\partial_t u + Lu + B(u) = h(t, x) + \chi(x)\eta(t, x), \quad (3.5)$$

where $\chi : \mathbb{T}^d \rightarrow \mathbb{R}_+$ is a smooth function, $h \in L^2_{\text{loc}}(\mathbb{R}_+; H^{s-1})$ is a given function, and η is a control taking values in a finite-dimensional subspace $\mathcal{H} \subset H^{s+2}$ that is specified below. For any $u_0 \in H^s$, $T > 0$, and $\zeta \in C(J_T; H^{s+1})$, let $\Theta(u_0, \zeta, T)$ be the set of controls $\eta \in L^2(J_T; H^{s-1})$ such that the problem (3.3), (3.2) has a unique solution in \mathcal{X}_T . From Proposition 3.1 it follows that the set $\Theta(u_0, \zeta, T)$ is open in $L^2(J_T; H^{s-1})$. Let \mathcal{R}_t be the restriction of the resolving operator \mathcal{R} at time $t \in J_T$.

As it is explained in the references [DR95, Hen78], one cannot control the trajectories of Eq. (3.5) outside the support of the function χ . We prove that the approximate controllability still holds if we restrict the problem to the interior $\mathcal{O} \subset \mathbb{T}^d$ of the level set Λ of χ given by

$$\Lambda = \{x \in \mathbb{T}^d : \chi(x) = M\}, \quad \text{where } M = \max_{x \in \mathbb{T}^d} \chi(x). \quad (3.6)$$

More precisely, we use the following notion of controllability.

Definition 3.2. Eq. (3.5) is said to be approximately controllable on the set \mathcal{O} in small time by \mathcal{H} -valued control if, for any $\varepsilon > 0$, any $T_0 > 0$, any $u_0 \in H^s$, and any $u_1 \in L^2$, there is a time $T \in (0, T_0)$ and a control $\eta \in \Theta(u_0, 0, T) \cap L^2(J_T; \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_0, 0, h + \chi\eta) - u_0 - u_1 \mathbb{I}_{\mathcal{O}}\|_{L^2} < \varepsilon,$$

where $\mathbb{I}_{\mathcal{O}}$ is the indicator function of the set \mathcal{O} .

Let $\mathcal{H} \subset H^{s+2}$ be a finite-dimensional subspace that is invariant under complex conjugation and contains the function identically equal to 1 on \mathbb{T}^d . Let us define a sequence of finite-dimensional subspaces $\{\mathcal{H}'_j\}$ of H^{s+2} by

$$\mathcal{H}'_0 = \mathcal{H}, \quad \mathcal{H}'_j = \text{span} \{B(\zeta) : \zeta \in \mathcal{H}'_{j-1}\}, \quad j \geq 1, \quad (3.7)$$

$$\mathcal{H}'_{\infty} = \cup_{j=0}^{\infty} \mathcal{H}'_j. \quad (3.8)$$

The proof of the below lemma is postponed to Section 4.2.

Lemma 3.3. *The following equality holds:*

$$\mathcal{H}'_j = \text{span} \{\zeta_1 \cdots \zeta_{2p+1} : \zeta_l \in \mathcal{H}'_{j-1}, l = 1, \dots, 2p+1\}, \quad j \geq 1. \quad (3.9)$$

As a consequence of this lemma, we see that the sequence $\{\mathcal{H}'_j\}$ is non-decreasing. We use the following notion of saturation in the case of the nonlinear CGL equation.

Definition 3.4. The subspace \mathcal{H} is \mathcal{O} -saturating for Eq. (3.5) if the subspace \mathcal{H}'_{∞} restricted to the set \mathcal{O} is dense in $L^2(\mathcal{O}; \mathbb{C})$.

The following is the main result of this section.

Theorem 3.5. *Assume that χ is such that $\mathcal{O} \neq \emptyset$, and \mathcal{H} is an \mathcal{O} -saturating subspace in the sense of Definition 3.4. Then Eq. (3.5) is approximately controllable on \mathcal{O} in small time by \mathcal{H} -valued control.*

By Proposition 4.2, the subspace $\mathcal{H}(\mathcal{K})$ defined by (0.4) and (0.6) is \mathbb{T}^3 -saturating in the sense of Definition 3.4. Hence, Theorem B given in the Introduction is obtained as a particular case of Theorem 3.5.

In the case when $\chi \equiv 1$ on \mathbb{T}^d and under a stronger saturation assumption, an approximate controllability property of usual form holds for Eq. (3.5).

Definition 3.6. Eq. (3.5) is said to be approximately controllable by \mathcal{H} -valued control if, for any $\varepsilon > 0$, any $T > 0$, and any $u_0, u_1 \in H^s$, there is a control $\eta \in \Theta(u_0, 0, T) \cap L^2(J_T; \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_0, 0, h + \eta) - u_1\|_s < \varepsilon.$$

Definition 3.7. The subspace \mathcal{H} is saturating for Eq. (3.5) if \mathcal{H}'_{∞} is dense in H^s .

Theorem 3.8. *Assume that $\chi \equiv 1$ on \mathbb{T}^d , and \mathcal{H} is a saturating subspace in the sense of Definition 3.7. Then Eq. (3.5) is approximately controllable by \mathcal{H} -valued control in the sense of Definition 3.6.*

A stronger version of this theorem holds when the subspace \mathcal{H} is of a special form. More precisely, let $\mathcal{I} \subset \mathbb{Z}^d$ be a finite set containing the zero vector, and let the subspace $\mathcal{H}(\mathcal{I})$ and the set $\tilde{\mathcal{I}} \subset \mathbb{Z}^d$ be defined as in the end of Section 1.2. Furthermore, let $H^s(\mathcal{I})$ be the closure in H^s of the subspace $\mathcal{H}(\tilde{\mathcal{I}})$.

Theorem 3.9. *Assume that $h \in L^2(J_T; H^{s-1}(\mathcal{I}))$. Then Eq. (3.5) is approximately controllable by $\mathcal{H}(\mathcal{I})$ -valued control in the sense of Definition 3.6 if and only if \mathcal{I} is a generator.*

Theorems 3.5, 3.8, and 3.9 are proved in the next subsection. The approximate controllability of nonlinear heat equations has been studied in the papers [FPZ95, FCZ00] when the control is localised in the physical space (but not in Fourier) and the nonlinear term grows slowly. The proof of the above three theorems is inspired by the approach of Agrachev and Sarychev introduced in the papers [AS05, AS06] to consider the approximate controllability of the 2D NS and Euler systems. That approach has been further extended and developed by many authors to various PDEs. See the papers [Shi06, Shi07, Ner15, Ner21a] for the study of the case of the 3D NS system, [Rod06, PR19] for the case of the NS system on rectangles with Lions boundary conditions, [Ner10, Ner11] for the 3D Euler system, [Sar12] for the 2D cubic Schrödinger equation, and [BGN20] for the 3D system of primitive equations of meteorology and oceanology. The arguments we use in the current setting are closer to the ones of the paper [Ner21b], where parabolic equation is considered with a polynomially growing nonlinearity.

In all the above papers, equations with additive controls are considered. Let us also mention the recent paper [DN21], where a version of Agrachev–Sarychev technics is proposed to study the controllability of the nonlinear Schrödinger equation with a multiplicative control.

3.2 Proof of the theorems

Let $\hat{\Theta}(u_0, T)$ be the set of pairs

$$(\eta, \zeta) \in L^2(J_T; H^{s-1}) \times C(J_T; H^{s+1})$$

such that the problem (3.3), (3.2) with $f = h + \chi\eta$ has a unique solution in \mathcal{X}_T . The following proposition plays an important role in our arguments.

Proposition 3.10. *For any $u_0, \eta \in H^{s+1}$, $\zeta \in H^{s+2}$, and $h \in L^2(J; H^{s-1})$, there is $\delta_0 > 0$ such that $(\delta^{-1}\eta, \delta^{-1/q}\zeta) \in \hat{\Theta}(u_0, \delta)$ for any $\delta \in (0, \delta_0)$, and the following limit holds:*

$$\mathcal{R}_\delta(u_0, \delta^{-1/q}\zeta, h + \delta^{-1}\chi\eta) \rightarrow u_0 + \chi\eta - B(\zeta) \quad \text{in } H^s \text{ as } \delta \rightarrow 0^+,$$

where $q = 2p + 1$.

This is proved in the same way as Proposition 2 in [Ner21b]; we shall not dwell on the details.

Proof of Theorem 3.5. To begin with, let us assume that $u_0 \in H^{s+1}$.

Step 1. Controllability to $u_0 + \chi\mathcal{H}_0$. Let us first note that the problem (3.5), (3.2) is approximately controllable in small time to any target in the set $u_0 + \chi\mathcal{H}_0$, i.e., for any $\varepsilon > 0$, $\eta \in \mathcal{H}_0$, and $T_0 > 0$, there are $T \in (0, T_0)$ and $\hat{\eta} \in \Theta(u_0, T) \cap L^2(J_T; \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_0, 0, h + \chi\hat{\eta}) - u_0 - \chi\eta\|_s < \varepsilon.$$

Indeed, this follows from Proposition 3.10 applied for the pair $(\eta, \zeta) = (\chi\eta, 0)$:

$$\mathcal{R}_\delta(u_0, 0, h + \delta^{-1}\chi\eta) \rightarrow u_0 + \chi\eta \quad \text{in } H^s \text{ as } \delta \rightarrow 0^+.$$

This implies the required property with $T = \delta$ and $\hat{\eta} = \delta^{-1}\eta$.

Step 2. Controllability to $u_0 + \chi^{q^N}\mathcal{H}_N$. Arguing by induction on $N \geq 0$, let us show that the problem (3.5), (3.2) is approximately controllable in small time to any target in $u_0 + \chi^{q^N}\mathcal{H}_N$. The base case $N = 0$ is considered in step 1. Assume that the property is proved for $N - 1$, and let $\eta \in \mathcal{H}_N$. Then, there are vectors $\zeta_1, \dots, \zeta_n \in \mathcal{H}_{N-1}$ such that

$$\eta = B(\zeta_1) + \dots + B(\zeta_n). \quad (3.10)$$

Applying Proposition 3.10 for the pair $(\eta, \zeta) = (0, \chi^{q^{N-1}}\zeta_1)$, we obtain

$$\mathcal{R}_\delta(u_0, \delta^{-1/q}\chi^{q^{N-1}}\zeta_1, h) \rightarrow u_0 - \chi^{q^N}B(\zeta_1) \quad \text{in } H^s \text{ as } \delta \rightarrow 0^+. \quad (3.11)$$

On the other hand, the following equality holds

$$\mathcal{R}_t(u_0 + \delta^{-1/q}\chi^{q^{N-1}}\zeta_1, 0, h) = \mathcal{R}_t(u_0, \delta^{-1/q}\chi^{q^{N-1}}\zeta_1, h) + \delta^{-1/q}\chi^{q^{N-1}}\zeta_1, \quad t \in J_\delta$$

by the uniqueness of the solution of the Cauchy problem. Taking in this equality $t = \delta$ and using (3.11), we get

$$\|\mathcal{R}_\delta(u_0 + \delta^{-1/q}\chi^{q^{N-1}}\zeta_1, 0, h) - u_0 + \chi^{q^N}B(\zeta_1) - \delta^{-1/q}\chi^{q^{N-1}}\zeta_1\|_s \rightarrow 0 \quad \text{as } \delta \rightarrow 0^+.$$

This limit, the assumption that $\zeta_1 \in \mathcal{H}_{N-1}$, the induction hypothesis, and Proposition 3.1 imply that there is a small time $T > 0$ and a control $\eta_1 \in \Theta(u_0, 0, T) \cap L^2(J_T; \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_0, 0, h + \chi\eta_1) - u_0 + \chi^{q^N}B(\zeta_1)\|_s < \varepsilon.$$

Iterating this argument for ζ_2, \dots, ζ_n , we construct a small time $\hat{T} > 0$ and a control $\hat{\eta} \in \Theta(u_0, 0, \hat{T}) \cap L^2(J_{\hat{T}}; \mathcal{H})$ such that (cf. (3.10))

$$\begin{aligned} & \|\mathcal{R}_{\hat{T}}(u_0, 0, h + \chi\hat{\eta}) - u_0 + \chi^{q^N}(B(\zeta_1) + \dots + B(\zeta_n))\|_s \\ &= \|\mathcal{R}_{\hat{T}}(u_0, 0, h + \chi\hat{\eta}) - u_0 + \chi^{q^N}\eta\|_s < \varepsilon. \end{aligned}$$

Thus, we have approximate controllability in small time to any target in the set $u_0 + \chi^{q^N} \mathcal{H}_N$.

Step 3. Conclusion. Without loss of generality, we can assume that the maximum M of the function χ (see (3.6)) equals to 1. Let us take any $u_1 \in L^2$. By the saturation hypothesis (Definition 3.4), there is an integer $N \geq 1$ and a vector $\eta \in \mathcal{H}_N$ such that

$$\|u_0 + \eta - \hat{u}_1\|_{L^2(\mathcal{O}; \mathbb{C})} < \varepsilon/2. \quad (3.12)$$

On the other hand, by the fact that the sequence $\{\mathcal{H}_j\}$ is non-decreasing and the results of steps 1 and 2, for any $\varepsilon > 0$ and $T_0 > 0$, there are sequences of integers $\{N_n\} \subset \mathbb{N}$, times $\{T_n\} \subset (0, T_0)$, and controls $\{\eta_n\} \subset \Theta(u_0, 0, T_n) \cap L^2(J_{T_n}, \mathcal{H})$ such that $N_n \rightarrow +\infty$ and

$$\|\mathcal{R}_{T_n}(u_0, 0, h + \chi\eta_n) - u_0 - \chi^{q^{N_n}} \eta\|_s < \varepsilon/2, \quad n \geq 1. \quad (3.13)$$

Combining this with (3.12), the fact that $\chi(x) \in [0, 1]$ for $x \in \mathbb{T}^2 \setminus \mathcal{O}$, and the Lebesgue theorem on dominated convergence, we derive approximate controllability in small time from the initial position $u_0 \in H^{s+1}$ to the target $u_0 + \mathbb{I}_{\mathcal{O}} u_1$ in the L^2 -norm. Taking η equal to zero on a small interval of time and using the regularising property of the CGL equation, we obtain approximate controllability in small time from arbitrary $u_0 \in H^s$. \square

Proof of Theorem 3.8. The starting point of the proof is the property (3.13), where we take $\chi \equiv 1$. The saturation assumption (Definition 3.7) implies that we have approximate controllability in small time in the sense that, for any $\varepsilon > 0$, $T_0 > 0$, and $u_0, u_1 \in H^s$, there are $T \in (0, T_0)$ and $\eta \in \Theta(u_0, 0, T) \cap L^2(J_T; \mathcal{H})$ such that

$$\|\mathcal{R}_T(u_0, 0, h + \eta) - u_1\|_s < \varepsilon.$$

Thus, the theorem will be proved if we show that, for any $\varepsilon > 0$, $T > 0$ and $u_1 \in H^s$, there is $\eta \in \Theta(u_1, 0, T) \cap L^2(J_T; \mathcal{H})$ verifying

$$\|\mathcal{R}_T(u_1, 0, h + \eta) - u_1\|_s < \varepsilon$$

(with initial condition coinciding with the target u_1). By Proposition 3.1, there are constants $r \in (0, \varepsilon)$ and $\tau > 0$ such that $(0, 0) \in \hat{\Theta}(v, \tau)$ and

$$\|\mathcal{R}_t(v, 0, h) - u_1\|_s < \varepsilon \quad \text{for any } v \in B_{H^s}(u_1, r), \quad t \in J_\tau.$$

If $\tau \geq T$, then the proof of the theorem is complete. If $\tau < T$, we use the approximate controllability property with initial condition $u_0 = \mathcal{R}_\tau(v, h)$, small time $T' < T - \tau$, and target u_1 . Thus, we find $\hat{\eta} \in \Theta(u_0, 0, T') \cap L^2(J_{T'}, \mathcal{H})$ such that

$$\mathcal{R}_{T'}(u_0, 0, h + \hat{\eta}) \in B_{H^s}(u_1, r).$$

Again, if $2\tau + T' > T$, then the proof is complete. If $2\tau + T' < T$, we apply the controllability property to return to $B_{H^s}(u_1, r)$. Iterating finitely many times this argument, we complete the proof. \square

Proof of Theorem 3.9. If \mathcal{I} is a generator, then $\mathcal{H}(\mathcal{I})$ is saturating by Proposition 4.2. Applying Theorem 3.8, we derive the required approximate controllability property.

If \mathcal{I} is not a generator, let l be any vector in the non-empty set $\mathbb{Z}^d \setminus \tilde{\mathcal{I}}$. The assumption that $h \in L^2(J_T; H^{s-1}(\mathcal{I}))$, the fact that $H^s(\mathcal{I})$ is invariant for the linear CGL equation (i.e., Eq. (3.5) with $B = 0$), and that the term B maps $H^s(\mathcal{I})$ to itself imply that the set

$$\mathcal{A} = \{\mathcal{R}_T(0, 0, h + \eta) : \eta \in \Theta(u_0, 0, T) \cap L^2(J_T; \mathcal{H})\}$$

is contained in $H^s(\mathcal{I})$. Thus, $\cos\langle l, x \rangle$ and $\sin\langle l, x \rangle$ are orthogonal to \mathcal{A} , so the set attainable from the origin with \mathcal{H} -valued controls is not dense in H^s . \square

4 Examples of saturating subspaces

4.1 Linearised equation

Let $s \geq 0$ be an arbitrary number, $\mathcal{I} \subset \mathbb{Z}^d$ be a finite set containing the zero vector, and $\mathcal{H}(\mathcal{I})$ be defined by (1.10). We denote by $\mathcal{H}_k(\mathcal{I})$, $k \in \mathbb{N} \cup \{\infty\}$ the subspaces \mathcal{H}_k given by relations (1.6) and (1.7) for $\mathcal{H} = \mathcal{H}(\mathcal{I})$. In this section, we prove the following result.

Proposition 4.1. *The subspace $\mathcal{H}_\infty(\mathcal{I})$ is dense in H^s if and only if \mathcal{I} is a generator.*

Proof. Assume that \mathcal{I} is a generator. From the identities

$$\begin{aligned} 2 \cos\langle l, x \rangle \cos\langle m, x \rangle &= \cos\langle l - m, x \rangle + \cos\langle l + m, x \rangle, \\ 2 \sin\langle l, x \rangle \sin\langle m, x \rangle &= \cos\langle l - m, x \rangle - \cos\langle l + m, x \rangle, \\ 2 \sin\langle l, x \rangle \cos\langle m, x \rangle &= \sin\langle l - m, x \rangle + \sin\langle l + m, x \rangle, \\ 2 \cos\langle l, x \rangle \sin\langle m, x \rangle &= \sin\langle m - l, x \rangle + \sin\langle l + m, x \rangle \end{aligned}$$

it follows that if $l, m \in \mathbb{Z}_*^d$ are such that

$$\cos\langle m, x \rangle, \sin\langle m, x \rangle \in \mathcal{H}(\mathcal{I}) \quad \text{and} \quad \cos\langle l, x \rangle, \sin\langle l, x \rangle \in \mathcal{H}_k(\mathcal{I})$$

for some $k \geq 0$, then

$$\cos\langle m \pm l, x \rangle, \sin\langle m \pm l, x \rangle \in \mathcal{H}_{k+1}(\mathcal{I}).$$

This implies that

$$\cos\langle l, x \rangle, \sin\langle l, x \rangle \in \mathcal{H}_\infty(\mathcal{I}) \quad \text{for any } l \in \mathbb{Z}^d,$$

so the space $\mathcal{H}_\infty(\mathcal{I})$ is dense in H^s for any $s \geq 0$.

If \mathcal{I} is not a generator, we argue as in the proof of Theorem 3.9. Let l be any vector in $\mathbb{Z}_*^d \setminus \tilde{\mathcal{I}}$. From the above trigonometric identities it follows that $\mathcal{H}_\infty(\mathcal{I}) \subset \mathcal{H}(\tilde{\mathcal{I}})$. Thus, the functions $\cos\langle l, x \rangle$ and $\sin\langle l, x \rangle$ are orthogonal to the subspace $\mathcal{H}_\infty(\mathcal{I})$ in H^s . Hence, $\mathcal{H}_\infty(\mathcal{I})$ is not dense in H^s . \square

4.2 Nonlinear equation

We start this section with a proof of Lemma 3.3.

Proof of Lemma 3.3. Let us denote by \mathcal{G} the subspace on the right-hand side of (3.9). Clearly, we have $\mathcal{H}'_j \subset \mathcal{G}$. To prove the inverse inclusion, we take any vectors $\zeta_l \in \mathcal{H}'_{j-1}$, $l = 1, \dots, 2p+1$ and consider the function $F : \mathbb{R}^{2p+1} \rightarrow \mathcal{H}'_j$ defined by

$$F(x_1, \dots, x_{2p+1}) = B \left(\sum_{l=1}^{2p+1} x_l \zeta_l \right).$$

As the subspace \mathcal{H}'_j is invariant under complex conjugation, we have $\operatorname{Re} \zeta \in \mathcal{H}'_j$ for any $\zeta \in \mathcal{H}'_j$. Let us choose $\zeta_l \in \mathcal{H}'_{j-1}$, $l = 1, \dots, 2p+1$ to be real vectors. Then

$$F(x_1, \dots, x_{2p+1}) = \left(\sum_{l=1}^{2p+1} x_l \zeta_l \right)^{2p+1}.$$

As the subspace \mathcal{H}'_j is closed, we have

$$\frac{\partial^{2p+1}}{\partial x_1 \dots \partial x_{2p+1}} F(0, \dots, 0) = (2p+1)! \zeta_1 \cdot \dots \cdot \zeta_{2p+1} \in \mathcal{H}'_j.$$

By linearity, this implies that $\zeta_1 \cdot \dots \cdot \zeta_{2p+1} \in \mathcal{H}'_j$ for any vectors $\zeta_l \in \mathcal{H}'_{j-1}$, $l = 1, \dots, 2p+1$. \square

Let $s \geq 0$ be arbitrary, $\mathcal{I} \subset \mathbb{Z}^d$ be a finite set containing zero, and $\mathcal{H}'_k(\mathcal{I})$, $k \in \mathbb{N} \cup \{\infty\}$ be the subspaces defined by (3.7) and (3.8) for $\mathcal{H} = \mathcal{H}(\mathcal{I})$.

Proposition 4.2. *The subspace $\mathcal{H}'_\infty(\mathcal{I})$ is dense in H^s if and only if \mathcal{I} is a generator.*

Proof. Let \mathcal{I} be a generator. From Lemma 3.3 and the assumption that $1 \in \mathcal{H}(\mathcal{I})$ it follows that $\mathcal{H}_j(\mathcal{I}) \subset \mathcal{H}'_j(\mathcal{I})$ for any $j \geq 1$. Thus, we have $\mathcal{H}_\infty(\mathcal{I}) \subset \mathcal{H}'_\infty(\mathcal{I})$, so that $\mathcal{H}'_\infty(\mathcal{I})$ is dense in H^s , by Proposition 4.1. The other assertion is proved as in Proposition 4.1, by noticing that $\mathcal{H}'_\infty(\mathcal{I}) \subset \mathcal{H}(\tilde{\mathcal{I}})$. \square

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