Large deviations for the Navier–Stokes equations driven by a white-in-time noise

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Abstract

In this paper, we consider the 2D Navier–Stokes system driven by a white-in-time noise. We show that the occupation measures of the trajectories satisfy a large deviations principle, provided that the noise acts directly on all Fourier modes. The proofs are obtained by developing an approach introduced previously for discrete-time random dynamical systems, based on a Kifer-type criterion and a multiplicative ergodic theorem.

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Contents

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0 Introduction

We study the large deviations principle (LDP) for the 2D Navier–Stokes system for incompressible fluids:

$$\partial_t u + \langle u, \nabla\rangle u - \nu\Delta u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \tag{0.1}$$

where $\nu > 0$ is the viscosity of the fluid, $u = (u_1(t,x), u_2(t,x))$ and $p = p(t,x)$ are unknown velocity field and pressure, f is an external (random) force, and $\langle u, \nabla\rangle = u_1\partial_1 + u_2\partial_2$. Throughout this paper, we assume that the space variable $x = (x_1, x_2)$ belongs [1] to the standard torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. The problem is considered in the space of divergence-free vector fields with zero mean value

$$H = \left\{u \in L^2(\mathbb{T}^2, \mathbb{R}^2) : \ \operatorname{div} u = 0 \text{ in } \mathbb{T}^2, \ \int_{\mathbb{T}^2} u(x)\mathrm{d}x = 0\right\} \tag{0.2}$$

endowed with the L^2-norm $\|\cdot\|$. We assume that the force is of the form

$$f(t,x) = h(x) + \eta(t,x),$$

where $h \in H^1 := H^1(\mathbb{T}^2, \mathbb{R}^2) \cap H$ is a given function and η is a white-in-time noise

$$\eta(t,x) = \partial_t W(t,x), \quad W(t,x) = \sum_{j=1}^\infty b_j\beta_j(t)e_j(x). \tag{0.3}$$

Here $\{b_j\}$ is sequence of real numbers such that

$$\mathfrak{B}_1 = \sum_{j=1}^\infty \alpha_j b_j^2 < \infty, \tag{0.4}$$

$\{\beta_j\}$ is a sequence of independent standard Brownian motions defined on a filtered probability space [2] $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, and $\{e_j\}$ is an orthonormal basis in H consisting of the eigenfunctions of the Stokes operator $L = -\Delta$ with eigenvalues $\{\alpha_j\}$. As usual, projecting (0.1) to H, we eliminate the pressure

[1] The periodic boundary conditions are chosen to simplify the presentation. Similar results can be established in the case of a bounded domain with smooth boundary and Dirichlet boundary conditions.

[2] We assume that this space satisfies the usual conditions (see Definition 2.25 in [15]).

and obtain an evolution equation for the velocity field[3] (e.g., see Section 6 in Chapter 1 of [21]):

$$\dot{u} + B(u) + Lu = h(x) + \eta(t, x), \tag{0.5}$$

where $B(u) = \Pi(\langle u, \nabla\rangle u)$ and Π is the orthogonal projection onto H in L^2. This system is supplemented with the initial condition

$$u(0) = u_0. \tag{0.6}$$

Under these assumptions, problem (0.5), (0.6) admits a unique solution and defines a Markov family $(u_t, \mathbb{P}_u)$ parametrised by the initial condition $u = u_0 \in H$. The ergodic properties of this family are now well understood. In particular, it is known that $(u_t, \mathbb{P}_u)$ admits a unique stationary measure, which is exponentially mixing, provided that sufficiently many coefficients b_j are non-zero (see the papers [7, 18, 6, 19, 1, 12, 24] and the book [20]). A central limit theorem (CLT) for problem (0.5), (0.6) is established in [16, 25]. The LDP proved in the present paper is a natural extension of the CLT. Indeed, while the CLT describes the probability of small deviations of a time average of a functional from its mean value, the LDP quantifies the probability of large deviations.

Before formulating the main result of this paper, let us introduce some notation and definitions. We denote by $\mathcal{P}(H)$ the space of Borel probability measures on H endowed with the topology of weak convergence. Given a measure $\nu \in \mathcal{P}(H)$, we set $\mathbb{P}_\nu(\Gamma) = \int_H \mathbb{P}_u(\Gamma)\nu(\mathrm{d}u)$ for any $\Gamma \in \mathcal{F}$ and consider the following family of *occupation measures*

$$\zeta_t = \frac{1}{t}\int_0^t \delta_{u_s} \mathrm{d}s, \quad t > 0 \tag{0.7}$$

defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_\nu)$. Here δ_u is the Dirac measure concentrated at $u \in H$. We shall say that a mapping $I : \mathcal{P}(H) \to [0, +\infty]$ is a *good rate function* if the level set $\{\sigma \in \mathcal{P}(H) : I(\sigma) \leq \alpha\}$ is compact for any $\alpha \geq 0$. A good rate function I is nontrivial if its effective domain $D_I = \{\sigma \in \mathcal{P}(H) : I(\sigma) < \infty\}$ is not a singleton. For any numbers $\varkappa > 0$ and $M > 0$, we denote

$$\Lambda(\varkappa, M) = \left\{\nu \in \mathcal{P}(H) : \int_H e^{\varkappa\|v\|^2}\nu(\mathrm{d}v) \leq M\right\}.$$

Main Theorem. *Assume that (0.4) holds and $b_j > 0$ for all $j \geq 1$. Then for any numbers $\varkappa > 0$ and $M > 0$, the family $\{\zeta_t, t > 0\}$ satisfies an LDP, uniformly with respect to $\nu \in \Lambda(\varkappa, M)$, with a non-trivial good rate function $I : \mathcal{P}(H) \to [0, +\infty]$ not depending on $\varkappa$ and M. More precisely, the following two bounds hold.*

Upper bound. *For any closed subset $F \subset \mathcal{P}(H)$, we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\nu\in\Lambda} \mathbb{P}_\nu\{\zeta_t \in F\} \leq -\inf_{\sigma\in F} I(\sigma).$$

[3]To simplify the notation, we shall assume that $\nu = 1$.

Lower bound. *For any open subset $G \subset \mathcal{P}(H)$, we have*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{\nu\in\Lambda} \mathbb{P}_\nu\{\zeta_t \in G\} \geq - \inf_{\sigma\in G} I(\sigma).$$

Furthermore, I is given by

$$I(\sigma) = \sup_{V\in C_b(H)} \left(\int_H V(u)\sigma(\mathrm{d}u) - Q(V) \right), \quad \sigma \in \mathcal{P}(H), \tag{0.8}$$

where $Q : C_b(H) \to \mathbb{R}$ is a 1-Lipschitz convex function such that $Q(C) = C$ for any $C \in \mathbb{R}$.

This type of large deviations results have been first established by Donsker and Varadhan [4] and later generalised by many others (see the books [9, 3, 2] and the references therein). There are only a few works studying the large deviations behaviour of solutions of randomly forced PDEs as time goes to infinity. The case of the stochastic Burgers and Navier–Stokes equations is first studied in [10, 11]. In these papers, the random perturbation is of the form (0.3) with the following restriction on the coefficients

$$cj^{-\alpha} \leq b_j \leq Cj^{-\frac{1}{2}-\varepsilon}, \quad \frac{1}{2} < \alpha < 1,\ \varepsilon \in \left(0, \alpha - \frac{1}{2}\right]. \tag{0.9}$$

Notice that the lower bound does not allow the sequence $\{b_j\}$ to converge to zero sufficiently fast, so the external force f is *irregular* with respect to the space variable. This is not very natural from the physical point of view. The proof is based on a general sufficient condition established in [26], and essentially uses the *strong Feller property*. The main novelty of our Main Theorem is that it proves an LDP without any lower bound on $\{b_j\}$ (so, in particular, we do not have a strong Feller property).

We use an approach introduced in the papers [13, 14], where an LDP is established for a family of dissipative PDEs perturbed by a *random kick force*. The proofs of these papers are based on a Kifer type criterion for LDP and a study of the large-time behaviour of generalised Markov semigroups. These results have been later extended in [22] to the case of the stochastic damped nonlinear wave equation driven by a *spatially regular white noise*. The main result of that paper is an LDP of local type. In the case of the Navier–Stokes system (0.5), although we follow a similar scheme, there are important differences in all the steps of the argument, coming from both the continuous-time nature of the system and the globalness of the LDP. Here we study the large-time asymptotics of the Feynman–Kac semigroup without any restriction on the smallness of the potential. One of the most important difficulties arises in the proof of the uniform Feller property. To establish this, we construct coupling processes using *a new two parameter auxiliary equation* (see (3.1)) which allows to have an appropriate Foiaş–Prodi estimate for the trajectories and a rapid exponential stabilisation for finite-dimensional projections.

Let us also mention that the multiplicative ergodic theorem we obtain for system (0.5) is of more general form and works for a larger class of functionals and initial measures (see Theorem 1.1).

It is a challenging *open problem* whether an LDP still holds for (0.5), (0.6) when the driving noise is *highly degenerate* (i.e., only a finite number of b_j are non-zero in (0.3)). For the Navier–Stokes system in this degenerate situation, exponential mixing is established in [12] for white-in-time noise and in [17] for a bounded noise satisfying some decomposability and observability hypotheses. Using these results and literally repeating the arguments of the proof of Theorem 5.4 in [23], one can prove a level-1 LDP of local type.

The paper is organised as follows. In Section 1, we state a multiplicative ergodic theorem for the Navier–Stokes system and combine it with Kifer's criterion for non-compact spaces to prove the Main Theorem. In Sections 2 and 3, we check the conditions of an abstract result on large-time behaviour of generalised Markov semigroups. Section 4 is devoted to the proof of the multiplicative ergodicity. In the Appendix, we prove various a priori estimates for the solutions and recall the statement of the above-mentioned result for generalised Markov semigroups.

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Notation

We shall use the following standard notation.

H is the space defined by (0.2), $B_H(a,R)$ is the closed ball in H of radius R centred at a. When $a=0$, we write $B_H(R)$.

$H^1 = H^1(\mathbb{T}^2,\mathbb{R}^2)\cap H$, where $H^1(\mathbb{T}^2,\mathbb{R}^2)$ is the space of vector functions $u=(u_1,u_2)$ with components in the usual Sobolev space of order 1 on $\mathbb{T}^2$.

$L^\infty(H)$ is the space of bounded Borel-measurable functions $f: H\to\mathbb{R}$ endowed with the norm $\|f\|_\infty = \sup_{u\in H}|f(u)|$.

$C_b(H)$ is the space of continuous functions $f\in L^\infty(H)$.

$L_b(H)$ is the space of functions $f\in C_b(H)$ for which the following norm is finite

$$\|\psi\|_L = \|\psi\|_\infty + \sup_{u\neq v}\frac{|\psi(u)-\psi(v)|}{\|u-v\|}.$$

$\mathcal{V}$ is the space of functions $V: H\to\mathbb{R}$ for which there is an integer $N\geq 1$ and a function $F\in L_b(H_N)$ such that

$$V(u) = F(\mathsf{P}_N u), \quad u\in H. \tag{0.10}$$

Here P_N is the orthogonal projection in H onto the space

$$H_N = \operatorname{span}\{e_1, \dots, e_N\} \tag{0.11}$$

and $\{e_j\}$ is the orthonormal basis entering (0.3).

For a given Borel-measurable function $\mathfrak{w} : H \to [1, +\infty]$, we denote by $C_{\mathfrak{w}}(H)$ (respectively, $L^\infty_{\mathfrak{w}}(H)$) the space of continuous (Borel-measurable) functions $f : H \to \mathbb{R}$ such that

$$\|f\|_{L^\infty_{\mathfrak{w}}} = \sup_{u \in H} \frac{|f(u)|}{\mathfrak{w}(u)} < \infty.$$

$\mathcal{M}_+(H)$ is the set of non-negative finite Borel measures on H endowed with the topology of the weak convergence. For $\mu \in \mathcal{M}_+(H)$ and $f \in C_b(H)$, we denote $\langle f, \mu \rangle = \int_H f(u)\mu(\mathrm{d}u)$.

$\mathcal{P}(H)$ is the set of Borel probability measures on H, and $\mathcal{P}_{\mathfrak{w}}(H)$ is the set of measures $\mu \in \mathcal{P}(H)$ such that $\langle \mathfrak{w}, \mu \rangle < \infty$.

1 Proof of the Main Theorem

In this section, we state a multiplicative ergodic theorem for the Navier–Stokes system (0.5) and apply it to prove the Main Theorem. Let us start by introducing the following two *weight functions*

$$\mathfrak{m}_\varkappa(u) = \exp(\varkappa \|u\|^2), \tag{1.1}$$

$$\mathfrak{w}_m(u) = 1 + \|u\|^{2m}, \quad u \in H \tag{1.2}$$

for positive numbers $\varkappa$ and m. To avoid triple subscripts, we shall write $C_{\mathfrak{m}}(H)$ and $\mathcal{P}_{\mathfrak{m}}(H)$ instead of $C_{\mathfrak{m}_\varkappa}(H)$ and $\mathcal{P}_{\mathfrak{m}_\varkappa}(H)$. Recall that the *Feynman–Kac semigroup* associated with the family $(u_t, \mathbb{P}_u)$ is defined by

$$\mathfrak{P}^V_t f(u) = \mathbb{E}_u \left\{ \Xi^V_t f(u_t) \right\},$$

where

$$\Xi^V_t = \exp\left(\int_0^t V(u_s) \mathrm{d}s \right). \tag{1.3}$$

From estimate (5.21) it follows that, for sufficiently small $\varkappa$ and any $V \in C_b(H)$, the application $\mathfrak{P}^V_t$ maps $C_{\mathfrak{m}}(H)$ into itself. Let $\mathfrak{P}^{V*}_t : \mathcal{M}_+(H) \to \mathcal{M}_+(H)$ be its dual. Then a measure $\mu \in \mathcal{P}(H)$ and a function $h \in C_{\mathfrak{m}}(H)$ are *eigenvectors* corresponding to an eigenvalue $\lambda > 0$ if

$$\mathfrak{P}^{V*}_t \mu = \lambda^t \mu, \qquad \mathfrak{P}^V_t h = \lambda^t h \quad \text{for any } t > 0.$$

We have the following result.

Theorem 1.1. *Under the conditions of the Main Theorem, for any $V \in \mathcal{V}$, there are numbers $m = m(V) \geq 1$ and $\gamma_0 = \gamma_0(\mathfrak{B}_0) > 0$, where $\mathfrak{B}_0 = \sum_{j \geq 1} b_j^2$, such that the following assertions hold for any $\varkappa \in (0, \gamma_0)$.*

Existence and uniqueness. *There is a unique pair* $(\mu_V, h_V) \in \mathcal{P}_{\mathfrak{m}}(H) \times C_{\mathfrak{w}}(H)$ *of eigenvectors corresponding to an eigenvalue* $\lambda_V > 0$ *normalised by the condition* $\langle h_V, \mu_V \rangle = 1$.

Convergence. *For any* $f \in C_{\mathfrak{m}}(H)$, $\nu \in \mathcal{P}(H)$, *and* $R > 0$, *we have*

$$\lambda_V^{-t} \mathfrak{P}_t^V f \to \langle f, \mu_V \rangle h_V \quad \text{in } C_b(B_H(R)) \cap L^1(H, \mu_V) \text{ as } t \to \infty, \tag{1.4}$$

$$\lambda_V^{-t} \mathfrak{P}_t^{V*} \nu \to \langle h_V, \nu \rangle \mu_V \quad \text{in } \mathcal{M}_+(H) \text{ as } t \to \infty. \tag{1.5}$$

Moreover, for any $M > 0$ *and* $\varkappa' \in (\varkappa, \gamma_0)$, *the convergence*

$$\lambda_V^{-t} \, \mathbb{E}_\nu \left\{ f(u_t) \exp\left(\int_0^t V(u_s) \mathrm{d}s \right) \right\} \to \langle f, \mu_V \rangle \, \langle h_V, \nu \rangle \quad \text{as } t \to \infty \tag{1.6}$$

holds uniformly in $\nu \in \Lambda(\varkappa', M)$.

This theorem is established in Section 4. Here we combine it with some arguments from [14, 22], to prove the Main Theorem.

Proof of the Main Theorem. Step 1: Reduction. It suffices to prove the Main Theorem for small $\varkappa$, so we shall assume that $\varkappa \in (0, \gamma_0)$. Let us take any $M > 0$ and endow the set

$$\Theta = \mathbb{R}_+^* \times \Lambda(\varkappa, M)$$

with an order relation $\prec$ defined by $(t_1, \nu_1) \prec (t_2, \nu_2)$ if and only if $t_1 \le t_2$. Then a family $\{x_\theta \in \mathbb{R}, \theta \in \Theta\}$ converges if and only if it converges uniformly with respect to $\nu \in \Lambda(\varkappa, M)$ as $t \to \infty$. Assume that the following three properties hold.

Property 1. For any $V \in C_b(H)$ and $\nu \in \Lambda(\varkappa, M)$, the following limit exists

$$Q(V) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_\nu \exp\left(\int_0^t V(u_s) \mathrm{d}s \right).$$

Moreover, it does not depend and is uniform in $\nu \in \Lambda(\varkappa, M)$.

Property 2. There is a vector space $\mathcal{V} \subset C_b(H)$ such that its restriction to any compact set $K \subset H$ is dense in $C(K)$, and for any $V \in \mathcal{V}$, there is a unique $\sigma_V \in \mathcal{P}(H)$ satisfying the relation

$$Q(V) = \langle V, \sigma_V \rangle - I(\sigma_V), \tag{1.7}$$

where $I(\sigma)$ is the Legendre transform of Q given by (0.8).

Property 3. There is a function $\Phi : H \to [0, +\infty]$ with compact level sets $\{u \in H : \Phi(u) \le \alpha\}$ for any $\alpha \ge 0$ such that

$$\mathbb{E}_\nu \exp\left(\int_0^t \Phi(u_s) \mathrm{d}s \right) \le C e^{ct}, \quad \nu \in \Lambda(\varkappa, M),\ t > 0 \tag{1.8}$$

for some positive constants C and c.

For any $\theta = (t, \nu) \in \Theta$, let us set $r_\theta := t$ and $\zeta_\theta := \zeta_t$, where ζ_t is the random probability measure given by (0.7) defined on the probability space $(\Omega_\theta, \mathcal{F}_\theta, \mathbb{P}_\theta) := (\Omega, \mathcal{F}, \mathbb{P}_\nu)$. The definition of the relation $\prec$ and Properties 1-3 imply that the family $\{\zeta_\theta\}$ satisfies the conditions of the Kifer type criterion established in Theorem 3.3 in [14]. Hence (0.8) defines a good rate function I and for any closed set $F \subset \mathcal{P}(H)$ and open set $G \subset \mathcal{P}(H)$, we have

$$\limsup_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in F\} \leq -\inf_{\sigma \in F} I(\sigma),$$
$$\liminf_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in G\} \geq -\inf_{\sigma \in G} I(\sigma).$$

These two inequalities imply the upper and lower bounds in the Main Theorem, since we have the following equalities

$$\limsup_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in F\} = \limsup_{t \to \infty} \frac{1}{t} \log \sup_{\nu \in \Lambda} \mathbb{P}_\nu\{\zeta_t \in F\},$$
$$\liminf_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in G\} = \liminf_{t \to \infty} \frac{1}{t} \log \inf_{\nu \in \Lambda} \mathbb{P}_\nu\{\zeta_t \in G\}.$$

Now we turn to the proofs of Properties 1-3.

Step 2: Proof of Properties 1-3. Property 3 is the easiest one. It is verified for $\varPhi(u) = \varkappa\|u\|_1^2$ if we choose $\varkappa \in (0, \gamma_0)$. Indeed, (1.8) follows from inequality [4] (5.20), and $\varPhi$ has compact level sets, since it is continuous from H^1 to $\mathbb{R}$ and the embedding $H^1 \subset H$ is compact.

Properties 1 and 2 are proved using the same methods as in the case of the discrete-time model considered in [14]. The restriction of $\mathcal{V}$ to any compact set $K \subset H$ is dense in $C(K)$. Taking $f = \mathbf{1}$ in (1.6), we get Property 1 for any $V \in \mathcal{V}$ with $Q(V) = \log \lambda_V$. In the case of an arbitrary $V \in C_b(H)$, this property is established by using a *buc-approximating* sequence $V_n \in \mathcal{V}$ of V (i.e., $\sup_{n \geq 1} \|V_n\|_\infty < \infty$ and $\|V_n - V\|_{L^\infty(K)} \to 0$ as $n \to \infty$ for any compact K in H) and the exponential tightness of the family $\{\zeta_\theta\}$ (which holds by Property 3). The reader is referred to Section 5.6 of [14] for the details.

To prove Property 2, for any $V \in \mathcal{V}$ and $F \in C_b(H)$, we consider the following auxiliary Markov semigroup

$$\mathscr{S}_t^{V,F} g(u) = \lambda_V^{-t} h_V^{-1} \mathfrak{P}_t^{V+F}(h_V g)(u), \quad g \in C_b(H),\, t \geq 0.$$

By Property 1, the following limit exists

$$Q^V(F) := \lim_{t \to \infty} \frac{1}{t} \log(\mathscr{S}_t^{V,F} \mathbf{1})(u).$$

Let $I^V : \mathcal{M}(H) \to [0, +\infty]$ be the Legendre transform of Q^V. The arguments of Section 5.7 in [14] show that $\sigma \in \mathcal{P}(H)$ satisfies (1.7) if and only if $I^V(\sigma) = 0$.

[4]We shall see in the proof of Theorem 1.1, that γ_0 is the number in Lemma 5.3.

On the other hand, by Proposition 1.3 in [22] (whose proof is the same in our case), the measure $\sigma_V = h_V \mu_V$ is the unique zero of I^V.

It remains to show that the good rate function I is non-trivial. Assume, by contradiction, that D_I is a singleton. Then $I(\mu) = 0$ and $I(\sigma) = +\infty$ for $\sigma \in \mathcal{P}(H) \setminus \{\mu\}$, where μ is the stationary measure of $(u_t, \mathbb{P}_u)$. On the other hand, as the Legendre transform is its own inverse, we derive from (0.8) that

$$Q(V) = \sup_{\sigma \in \mathcal{P}(H)} \big(\langle V, \sigma\rangle - I(\sigma)\big) \quad \text{for } V \in C_b(H).$$

This implies that $Q(V) = \langle V, \mu\rangle$ for any $V \in C_b(H)$. Let us take any non-constant $V \in \mathcal{V}$ such that $\langle V, \mu\rangle = 0$. Then $Q(V) = 0$, and from limit (1.4) with $f = \mathbf{1}$ and $\nu = \mu$ we get $\lambda_V = e^{Q(V)} = 1$ and

$$\sup_{t\geq 0} \mathbb{E}_\mu \exp\left(\int_0^t V(u_s)\mathrm{d}s\right) < \infty. \tag{1.9}$$

Combining the latter with the central limit theorem (see Proposition 4.1.4 in [20]), we get $V = \mathbf{0}$. This contradicts the assumption that V is non-constant and completes the proof of the Main Theorem.

$\square$

2 Checking conditions of Theorem 5.6

Theorem 1.1 is proved by applying a convergence result for generalised Markov semigroups obtained in [14, 22] and restated here as Theorem 5.6. In this and next sections, we show that the conditions of that theorem are satisfied for the generalised Markov family of transition kernels defined by

$$P_t^V(u, \Gamma) = (\mathfrak{P}_t^{V*}\delta_u)(\Gamma), \quad \Gamma \in \mathcal{B}(H),\, u \in H,$$

if we take $X = H$, $X_R = B_{H^1}(R)$, and $\mathfrak{w} = \mathfrak{w}_m$ with sufficiently large $m \geq 1$.

2.1 Growth estimates

Estimate (5.24) implies that the measure $P_t^V(u, \cdot)$ is concentrated on the space $H^1 = \cup_{R=1}^\infty X_R = X_\infty$ for any $V \in C_b(H)$, $t > 0$, and $u \in H$. The boundedness of V implies that $\sup_{t\in[0,1]} \|\mathfrak{P}_t^V \mathbf{1}\|_\infty < \infty$. So the following proposition gives the growth condition in Theorem 5.6.

Proposition 2.1. *For any $V \in C_b(H)$, there are positive numbers m and R_0 such that*

$$\sup_{t\geq 0} \frac{\|\mathfrak{P}_t^V \mathfrak{w}\|_{L^\infty_\mathfrak{w}}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}} < \infty, \tag{2.1}$$

where $\mathfrak{w} = \mathfrak{w}_m$ and $\|\cdot\|_{R_0}$ is the L^∞ norm on X_{R_0}.

Proof. Replacing V by $V - \inf_H V$, we can assume that V is non-negative.

Step 1. Let us show that there are integers $m, R_0 \geq 1$ such that

$$\sup_{t\geq 0} \frac{\|\mathfrak{P}_t^V \mathbf{1}\|_{L_{\mathfrak{w}}^\infty}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}} < \infty. \tag{2.2}$$

Indeed, let $\tau(R)$ be the first hitting time of the set X_R defined by (5.26), and let m and R_0 be the integers in Proposition 5.4 for $\gamma = \|V\|_\infty$. Then for any $u \in H$, we have

$$\mathfrak{P}_t^V \mathbf{1}(u) = \mathbb{E}_u \Xi_t^V = \mathbb{E}_u \big\{ \mathbb{I}_{G_t} \Xi_t^V \big\} + \mathbb{E}_u \big\{ \mathbb{I}_{G_t^c} \Xi_t^V \big\} =: I_1 + I_2, \tag{2.3}$$

where Ξ_t^V is given by (1.3) and $G_t = \{\tau(R_0) > t\}$. As V is non-negative, we have $\mathfrak{P}_t^V \mathbf{1}(u) \geq 1$. This and (5.27) imply that

$$I_1 \leq \mathbb{E}_u \Xi_{\tau(R_0)}^V \leq \mathbb{E}_u \exp\big(\gamma \tau(R_0)\big) \leq C\, \mathfrak{w}(u) \leq C\, \mathfrak{w}(u)\, \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}. \tag{2.4}$$

By the strong Markov property and (5.27),

$$\begin{aligned} I_2 &\leq \mathbb{E}_u \big\{ \mathbb{I}_{G_t} \Xi_{\tau(R_0)}^V \mathbb{E}_{u(\tau(R_0))} \Xi_t^V \big\} \\ &\leq \mathbb{E}_u \{ e^{\gamma \tau(R_0)} \}\, \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \leq C\, \mathfrak{w}(u)\, \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}. \end{aligned} \tag{2.5}$$

Inequalities (2.3)-(2.5) imply (2.2).

Step 2. It suffices to prove (2.1) for integer times $k \geq 1$:

$$\sup_{k\geq 0} \frac{\|\mathfrak{P}_k^V \mathfrak{w}\|_{L_{\mathfrak{w}}^\infty}}{\|\mathfrak{P}_k^V \mathbf{1}\|_{R_0}} < \infty. \tag{2.6}$$

Indeed, the semigroup property and the fact that V is non-negative and bounded imply that

$$\begin{aligned} &\|\mathfrak{P}_t^V \mathfrak{w}\|_{L_{\mathfrak{w}}^\infty} = \|\mathfrak{P}_{t-[t]}^V (\mathfrak{P}_{[t]}^V \mathfrak{w})\|_{L_{\mathfrak{w}}^\infty} \leq C_0 \|\mathfrak{P}_{[t]}^V \mathfrak{w}\|_{L_{\mathfrak{w}}^\infty}, \\ &\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \geq \|\mathfrak{P}_{[t]}^V \mathbf{1}\|_{R_0}, \end{aligned}$$

where $[t]$ is the integer part of t and $C_0 := \sup_{s\in[0,1]} \|\mathfrak{P}_s^V \mathfrak{w}\|_{L_{\mathfrak{w}}^\infty}$. By (5.23), we have

$$C_0 \leq e^\gamma \sup_{s\in[0,1]} \|\mathfrak{P}_s \mathfrak{w}\|_{L_{\mathfrak{w}}^\infty} < \infty,$$

where $\mathfrak{P}_t = \mathfrak{P}_t^0$ is the Markov operator associated with (0.5).

Step 3. To prove (2.6), we use the Markov property and (5.23):

$$\begin{aligned} \mathfrak{P}_k^V \mathfrak{w}(u) &\leq e^\gamma \mathbb{E}_u \left\{ \Xi_{k-1}^V \mathfrak{w}(u_k) \right\} \\ &= e^\gamma \mathbb{E}_u \left\{ \Xi_{k-1}^V \mathbb{E}_{u_{k-1}} \mathfrak{w}(u_1) \right\} \\ &\leq e^\gamma \mathbb{E}_u \left\{ \Xi_{k-1}^V [e^{-m\alpha_1} \mathfrak{w}(u_{k-1}) + C_1] \right\} \\ &\leq q \mathfrak{P}_{k-1}^V \mathfrak{w}(u) + e^\gamma C_1 \mathfrak{P}_{k-1}^V \mathbf{1}(u), \end{aligned}$$

where we choose $m > \gamma/\alpha_1$, so that $q := e^{\gamma - m\alpha_1} < 1$. Iterating this inequality and using the fact that the sequence $\{\|\mathfrak{P}_k^V \mathbf{1}\|_{R_0}\}$ is a non-decreasing in k, we obtain

$$\mathfrak{P}_k^V \mathfrak{w}(u) \le q^k \mathfrak{w}(u) + (1-q)^{-1} e^{\gamma} C_1 \mathfrak{P}_k^V \mathbf{1}(u).$$

This and (2.2) imply (2.6).

□

We shall also need the following growth estimates with two other weights.

Proposition 2.2. *Let $V \in C_b(H)$ and let R_0 and γ_0 be the numbers in Proposition 2.1 and Lemma 5.3, respectively. Then for any $\varkappa \in (0, \gamma_0)$, we have*

$$\sup_{t\ge 0} \frac{\|\mathfrak{P}_t^V \mathfrak{m}\|_{L_{\mathfrak{m}}^\infty}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}} < \infty, \tag{2.7}$$

$$\sup_{t\ge 1} \frac{\|\mathfrak{P}_t^V F\|_{L_{\mathfrak{m}}^\infty}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}} < \infty, \tag{2.8}$$

where $\mathfrak{m} = \mathfrak{m}_\varkappa$ and $F(u) = \|u\|_1^2$.

Proof. Step 1: Proof of (2.7). As in the previous proof, we can assume that V is non-negative and $t = k$ is integer. We take any $A > 0$ and write

$$\begin{aligned}\mathfrak{P}_k^V \mathfrak{m}(u) &= \mathbb{E}_u \left\{\mathbb{I}_{\{\|u_k\|^2 \le A\}} \Xi_k^V \mathfrak{m}(u_k)\right\} + \mathbb{E}_u \left\{\mathbb{I}_{\{\|u_k\|^2 > A\}} \Xi_k^V \mathfrak{m}(u_k)\right\} \\ &=: I_k + J_k.\end{aligned} \tag{2.9}$$

By (2.2), we have

$$\|\mathfrak{P}_k^V \mathbf{1}\|_{L_{\mathfrak{m}}^\infty} \le C_2 \|\mathfrak{P}_k^V \mathbf{1}\|_{R_0},$$

hence

$$\|I_k\|_{L_{\mathfrak{m}}^\infty} \le e^{\varkappa A} \|\mathfrak{P}_k^V \mathbf{1}\|_{L_{\mathfrak{m}}^\infty} \le C_2 e^{\varkappa A} \|\mathfrak{P}_k^V \mathbf{1}\|_{R_0}. \tag{2.10}$$

To estimate J_k, we use the Markov property and (5.22)

$$\begin{aligned}J_k(u) &\le A^{-1} \mathbb{E}_u \left\{\|u_k\|^2 \Xi_k^V \mathfrak{m}(u_k)\right\} \le A^{-1} e^{\gamma}\, \mathbb{E}_u \left\{\|u_k\|^2 \Xi_{k-1}^V \mathfrak{m}(u_k)\right\} \\ &= A^{-1} e^{\gamma}\, \mathbb{E}_u \left\{\Xi_{k-1}^V \mathbb{E}_{u_{k-1}} \left\{\|u_1\|^2 \mathfrak{m}(u_1)\right\}\right\} \le A^{-1} C_3 \mathfrak{P}_{k-1}^V \mathfrak{m}(u).\end{aligned}$$

Combining this with (2.9) and (2.10), and choosing $A > 0$ so large that $q := A^{-1} C_3 < 1$, we get

$$\|\mathfrak{P}_k^V \mathfrak{m}\|_{L_{\mathfrak{m}}^\infty} \le C_2 e^{\varkappa A} \|\mathfrak{P}_k^V \mathbf{1}\|_{R_0} + q \|\mathfrak{P}_{k-1}^V \mathfrak{m}\|_{L_{\mathfrak{m}}^\infty}.$$

Iterating, we obtain

$$\|\mathfrak{P}_k^V \mathfrak{m}\|_{L_{\mathfrak{m}}^\infty} \le C_2 e^{\varkappa A} (1-q)^{-1} \|\mathfrak{P}_k^V \mathbf{1}\|_{R_0} + q^k.$$

As $\mathfrak{P}_k^V \mathbf{1}(u) \ge 1$, we arrive at the required inequality (2.7).

Step 2: Proof of (2.8). For any $t \geq 1$, we have

$$\mathfrak{P}_t^V F = \mathfrak{P}_{t-1}^V(\mathfrak{P}_1^V F) \leq e^{\gamma}\, \mathfrak{P}_{t-1}^V(\mathfrak{P}_1 F).$$

So (5.24) and (2.7) imply that

$$\mathfrak{P}_t^V F(u) \leq C_4 \mathfrak{P}_{t-1}^V \mathfrak{w}_8(u) \leq C_5 \mathfrak{P}_{t-1}^V \mathfrak{m}(u) \leq C_6 \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \mathfrak{m}(u).$$

This proves (2.8). ☐

2.2 Time-continuity

The following lemma proves the time-continuity property.

Lemma 2.3. *The function $t \mapsto \mathfrak{P}_t^V g(u)$ is continuous from $\mathbb{R}_+$ to $\mathbb{R}$ for any $V \in C_b(H)$, $g \in C_{\mathfrak{w}}(H)$, $u \in H$, and $\mathfrak{w} = \mathfrak{w}_m$ with any $m \geq 1$.*

Proof. Let us show the continuity at the point $T \geq 0$. For any $t \geq 0$, we write

$$\begin{aligned}\mathfrak{P}_T^V g(u) - \mathfrak{P}_t^V g(u) &= \mathbb{E}_u \left\{ \left[\Xi_T^V - \Xi_t^V\right] g(u_t) \right\} + \mathbb{E}_u \left\{ \left[g(u_T) - g(u_t)\right] \Xi_T^V \right\} \\ &=: S_1 + S_2.\end{aligned}$$

As V is bounded and $g \in C_{\mathfrak{w}}(H)$, we have

$$\begin{aligned}|S_1| &\leq \mathbb{E}_u \left\{ \left| \exp\left(\int_t^T V(u_s) \mathrm{d}s \right) - 1 \right| \Xi_t^V |g(u_t)| \right\} \\ &\leq \|g\|_{L_{\mathfrak{w}}^\infty} \left(e^{|T-t| \|V\|_\infty} - 1 \right) e^{T\|V\|_\infty} \mathbb{E}_u \mathfrak{w}(u_t).\end{aligned}$$

Combining this with (5.23), we get $S_1 \to 0$ as $t \to T$. To estimate S_2, we take any $R > 0$ and write

$$\begin{aligned}e^{-T\|V\|_\infty} |S_2| &\leq \mathbb{E}_u \left| g(u_T) - g(u_t) \right| \\ &= \mathbb{E}_u \left\{ \mathbb{I}_{G_R^c} \left| g(u_T) - g(u_t) \right| \right\} + \mathbb{E}_u \left\{ \mathbb{I}_{G_R} \left| g(u_T) - g(u_t) \right| \right\} \\ &=: S_3 + S_4,\end{aligned}$$

where $G_R := \{u_t, u_T \in B_H(R)\}$. From $g \in C_{\mathfrak{w}}(H)$ and (5.23) we derive

$$\begin{aligned}S_3 &\leq C_1 \mathbb{E}_u \left\{ \mathbb{I}_{G_R^c} (\mathfrak{w}(u_T) + \mathfrak{w}(u_t)) \right\} \\ &\leq C_1 R^{-1} \mathbb{E}_u \left\{ \mathfrak{w}^2(u_T) + \mathfrak{w}^2(u_t) \right\} \leq C_2 R^{-1} \mathfrak{w}^2(u).\end{aligned}$$

On the other hand, by the Lebesgue theorem on dominated convergence, for any $R > 0$, we have $S_4 \to 0$ as $t \to T$. Choosing $R > 0$ sufficiently large and t sufficiently close to T, we see that $S_3 + S_4$ can be made arbitrarily small. This shows that $S_2 \to 0$ as $t \to T$ and completes the proof of the lemma. ☐

2.3 Uniform irreducibility

As V is a bounded function, we have

$$P_t^V(u, \mathrm{d}v) \geq e^{-t\|V\|_\infty} P_t(u, \mathrm{d}v), \quad u \in H,$$

where $P_t(u, \cdot)$ is the transition function of the Markov family $(u_t, \mathbb{P}_u)$. Thus to show the uniform irreducibility of $\{P_t^V\}$, it suffices to prove the following result.

Proposition 2.4. *The family $\{P_t\}$ is uniformly irreducible with respect to the sequence $\{X_R\}$, i.e., for any integers $\rho, R \geq 1$ and any $r > 0$, there are positive numbers $l = l(\rho, r, R)$ and $p = p(\rho, r)$ such that*

$$P_l(u, B_H(\hat{u}, r)) \geq p, \quad u \in B_H(R), \, \hat{u} \in X_\rho. \tag{2.11}$$

Proof. Step 1. There is a number $d > 0$ such that for any $R \geq 1$, we have

$$P_t(u, X_d) \geq \frac{1}{2}, \quad u \in B_H(R) \tag{2.12}$$

for sufficiently large $t = t(R)$. Indeed, combining (5.23), (5.24), and the Markov property, we get

$$\mathbb{E}_u \|u_t\|_1^2 \leq C(e^{-8\alpha_1 t} R^8 + 1), \quad u \in B_H(R), \, t \geq 1.$$

Taking t so large that $e^{-8\alpha_1 t} R^8 < 1$ and $d > 2\sqrt{C}$ and using the Chebyshev inequality, we arrive at

$$P_t(u, X_d) \geq 1 - d^{-2} C(e^{-8\alpha_1 t} R^8 + 1) \geq \frac{1}{2}.$$

Step 2. By Lemma 3.3.11 in [20], for any non-degenerate ball $B \subset H$, there is $p_1 = p_1(d, B) > 0$ such that

$$P_1(u, B) \geq p_1, \quad u \in X_d.$$

Combining this with a simple compactness and continuity argument, we get

$$P_1(u, B_H(\hat{u}, r)) \geq p_2, \quad u \in X_d, \, \hat{u} \in X_\rho$$

for some $p_2 = p_2(d, \rho, r) > 0$. This estimate, (2.12), and the Kolmogorov–Chapman relation imply (2.11) with $l = t + 1$ and $p = p_2/2$. ☐

2.4 Existence of an eigenvector

Here we show that the dual operator $\mathfrak{P}_t^{V*}$ has an eigenvector and give some decay estimates for it.

Proposition 2.5. *For any $V \in C_b(H)$ and $t > 0$, the operator $\mathfrak{P}_t^{V*}$ has at least one eigenvector $\mu_{t,V} \in \mathcal{P}(H)$ with a positive eigenvalue $\lambda_{t,V}$:*

$$\mathfrak{P}_t^{V*}\mu_{t,V} = \lambda_{t,V}\mu_{t,V}. \tag{2.13}$$

Moreover, any such eigenvector satisfies

$$\int_H (\|u\|_1^n + \mathfrak{m}_\varkappa(u))\,\mu_{t,V}(\mathrm{d}u) < \infty, \tag{2.14}$$

$$\|\mathfrak{P}_t^V \mathfrak{w}_m\|_{X_R} \int_{X_R^c} \mathfrak{w}_m(u)\mu_{t,V}(\mathrm{d}u) \to 0 \quad \text{as } R \to \infty \tag{2.15}$$

for any $\varkappa \in (0, \gamma_0)$ and $n, m \geq 1$.

Proof. Step 1: Estimate (2.14). Let us fix $t > 0$, and let $\mu \in \mathcal{P}(H)$ be an eigenvector of the operator $\mathfrak{P}_t^{V*}$ corresponding to an eigenvalue $\lambda > 0$. Let us show that $\mu \in \mathcal{P}_\mathfrak{m}(H)$ with $\mathfrak{m} = \mathfrak{m}_\varkappa$ for any $\varkappa \in (0, \gamma_0)$. Indeed, for any measurable function $f : H \to \mathbb{R}_+ \cup \{+\infty\}$, we have

$$\langle f, \mu\rangle = \lambda^{-1}\langle \mathfrak{P}_t^V f, \mu\rangle \leq \lambda^{-1} e^{t\|V\|_\infty}\langle \mathfrak{P}_t f, \mu\rangle. \tag{2.16}$$

Taking $f = \mathfrak{m}_\varkappa$, any number $A > 0$, and setting[5] $C_1 = \lambda^{-1}e^{t\|V\|_\infty}$, we obtain

$$\begin{aligned}
\int_H \mathfrak{m}_\varkappa(u)\mu(\mathrm{d}u) &\leq C_1 \int_H \mathbb{E}_u\{\mathfrak{m}_\varkappa(u_t)\}\mu(\mathrm{d}u) \\
&= C_1 \int_H \Big(\mathbb{E}_u\left\{\mathbb{I}_{\{\|u_t\|^2 \leq A\}}\mathfrak{m}_\varkappa(u_t)\right\} + \mathbb{E}_u\left\{\mathbb{I}_{\{\|u_t\|^2 > A\}}\mathfrak{m}_\varkappa(u_t)\right\}\Big)\mu(\mathrm{d}u) \\
&\leq C_1 \int_H \left(\exp(\varkappa A) + A^{-1}\mathbb{E}_u\left\{\|u_t\|^2\mathfrak{m}_\varkappa(u_t)\right\}\right)\mu(\mathrm{d}u) \\
&\leq C_1 \int_H \left(\exp(\varkappa A) + C_2 A^{-1}\mathfrak{m}_\varkappa(u)\right)\mu(\mathrm{d}u),
\end{aligned}$$

where we used inequality (5.22). Choosing $A > C_1C_2$, we get

$$\int_H \mathfrak{m}_\varkappa(u)\mu(\mathrm{d}u) \leq C_1(1 - C_1C_2A^{-1})^{-1}\exp(\varkappa A) < \infty, \tag{2.17}$$

so[6] $\mu \in \mathcal{P}_\mathfrak{m}(H)$. Taking $f(u) = \|u\|_1^n$ in (2.16) and using (5.24) and (2.17), we obtain

$$\int_H \|u\|_1^n \mu(\mathrm{d}u) \leq C_1 \int_H \mathbb{E}_u\{\|u_t\|_1^n\}\mu(\mathrm{d}u) \leq C_3 \int_H (1 + \|u\|^{8n})\mu(\mathrm{d}u) < \infty$$

for any $n \geq 1$. This proves (2.14).

[5] We do not indicate the dependence of different constants on V, t, m, n, and $\varkappa$.

[6] Note that this proof is formal. A rigorous proof can be obtained by applying the above arguments to bounded approximations of $\mathfrak{m}$.

Step 2: Limit (2.15). From (5.23) it follows that

$$\begin{aligned}\|\mathfrak{P}_t^V \mathfrak{w}_m\|_{X_R} &\le e^{t\|V\|_\infty} \sup_{u\in X_R} \mathbb{E}_u \mathfrak{w}_m(u_t)\\ &\le C_4 \sup_{u\in B_H(R)} \mathfrak{w}_m(u) = C_4(1+R^{2m}).\end{aligned} \tag{2.18}$$

Using the Cauchy–Schwarz inequality, (2.14), and the Chebyshev inequality, we see that

$$\int_{X_R^c} \mathfrak{w}_m(u)\mu(\mathrm{d}u) \le \langle \mathfrak{w}_m^2, \mu\rangle^{1/2} \mu\big(X_R^c\big)^{1/2} \le C_5 R^{-n}.$$

Combining this with (2.18) and choosing $n > 2m$, we obtain (2.15).

Step 3: Construction of an eigenvector. Let us take any $A > 0$ and $m \ge 1$ and define the convex set

$$D_{A,m} := \big\{\nu \in \mathcal{P}(H) : \langle \mathfrak{w}_m, \nu\rangle \le A\big\}.$$

By the Fatou lemma, $D_{A,m}$ is closed in $\mathcal{P}(H)$. Consider the continuous mapping

$$G := G(t,V) : D_{A,m} \to \mathcal{P}(H), \qquad \nu \mapsto \frac{\mathfrak{P}_t^{V*}\nu}{\mathfrak{P}_t^{V*}\nu(H)}.$$

Let us show that $G(D_{A,m}) \subset D_{A,m}$ for an appropriate choice of A and m, and that $G(D_{A,m})$ is compact in $\mathcal{P}(H)$. In view of the Leray–Schauder theorem, this will imply the existence of an eigenvector $\mu \in D_{A,m}$ satisfying (2.13) with eigenvalue $\lambda = \mathfrak{P}_t^{V*}\mu(H) > 0$. From (5.23) we derive that

$$\begin{aligned}\big\langle \mathfrak{w}_m, G(\nu)\big\rangle &\le \exp\{t \operatorname{Osc}(V)\}\langle \mathfrak{w}_m, \mathfrak{P}_t^*\nu\rangle\\ &\le \exp\{t(\operatorname{Osc}(V) - m\alpha_1)\}\langle \mathfrak{w}_m, \nu\rangle + C_6,\end{aligned}$$

where $\operatorname{Osc}(V) := \sup_{u\in H} V(u) - \inf_{u\in H} V(u)$ is the oscillation of V. Choosing A and m so large that $\exp\{t(\operatorname{Osc}(V) - m\alpha_1)\} \le 1/2$ and $A \ge 2C_6$, we get that $G(D_{A,m}) \subset D_{A,m}$. In view of the Prokhorov compactness criterion (see Theorem 11.5.4 in [5]), to prove that $G(D_{A,m})$ is relatively compact, it suffices to check that

$$\int_H \|u\|_1^2 \mathfrak{P}_t^{V*}\nu(\mathrm{d}u) \le C_7 \quad \text{for any } \nu \in D_{A,m}.$$

Using (5.24) and the fact that V is bounded, we get

$$\begin{aligned}\int_H \|u\|_1^2 \mathfrak{P}_t^{V*}\nu(\mathrm{d}u) &\le \exp(t\|V\|_\infty) \int_H \|u\|_1^2 (\mathfrak{P}_t^*\nu)(\mathrm{d}u)\\ &\le C_8 \int_H \|u\|^8 \nu(\mathrm{d}u)\\ &\le C_9 \int_H \mathfrak{w}_m(u)\nu(\mathrm{d}u) \le C_9 A =: C_7.\end{aligned}$$

Thus there is an eigenvector $\mu \in D_{A,m}$.

$\square$

3 Uniform Feller property

In this section, we establish the following result.

Theorem 3.1. *For any $V \in \mathcal{V}$, the family $\{P_t^V\}$ satisfies the uniform Feller property with respect to the sequence $\{X_R\}$, i.e., there is an integer $R_0 \geq 1$ such that the family $\{\|\mathfrak{P}_t^V \mathbf{1}\|_R^{-1} \mathfrak{P}_t^V \psi, t \geq 0\}$ is uniformly equicontinuous on X_R for any $\psi \in \mathcal{V}$ and $R \geq R_0$.*

See the papers [13, 14, 23] for similar results in the case of a discrete-time random dynamical system and [22] for the case of the stochastic damped nonlinear wave equation. The main difficulty in the proof of Theorem 3.1 comes from the fact that the oscillation of the potential V can be arbitrarily large. To overcome this, we introduce a new auxiliary equation in the construction of the coupling processes and choose carefully the parameters in order to have a stabilisation property with an appropriate rate.

3.1 Construction of coupling processes

The coupling processes are constructed following the arguments of [22]. Let us take any $z, z' \in H$ and denote by u_t and u_t' the solutions of (0.5) issued from z and z'. For any integer $N \geq 1$ and number $\lambda > 0$, let v be the solution of the following problem

$$\dot{v} + B(v) + Lv + \mathsf{P}_N[\lambda(v - u) + B(u) - B(v)] = h + \eta(t), \quad v(0) = z', \tag{3.1}$$

where η is defined by (0.3). We denote by $\nu(z, z')$ and $\nu'(z')$ the laws of processes $\{v(t), t \in J\}$ and $\{u'(t), t \in J\}$, respectively, where $J = [0, 1]$. We shall use the following result.

Proposition 3.2. *There exists an integer $N_1 \geq 1$ such that if $N \geq N_1$ and $\lambda \geq N^2/2$, then for any $\varepsilon > 0$ and $z, z' \in H$, we have*

$$\|\nu(z, z') - \nu'(z')\|_{var} \leq \varepsilon^a + 2\left[\exp\left(C_{\lambda,N}\varepsilon^{a-2}\|z - z'\|^2 e^{C(\|z\|^2 + \|z'\|^2)}\right) - 1\right]^{1/2}, \tag{3.2}$$

where $\|\cdot\|_{var}$ denotes the total variation distance on $\mathcal{P}(C(J; H))$ and $a < 2, C$, and $C_{\lambda,N}$ are positive constants not depending on ε, z, z'.

See Section 5.2 for the proof. By Proposition 1.2.28 in [20], there is a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and measurable functions $\mathcal{Z}, \mathcal{Z}' : H \times H \times \hat{\Omega} \to C(J; H)$ such that $(\mathcal{Z}(z, z'), \mathcal{Z}'(z, z'))$ is a maximal coupling for $(\nu(z, z'), \nu(z'))$ for any $z, z' \in H$. We denote by $\tilde{v}$ and $\tilde{u}_t'$ the restrictions of $\mathcal{Z}$ and $\mathcal{Z}'$ to time $t \in J$. Then $\tilde{v}_t$ is a solution of

$$\dot{\tilde{v}} + B(\tilde{v}) + L\tilde{v} + \mathsf{P}_N[\lambda\tilde{v} - B(\tilde{v})] = h + \psi(t), \quad \tilde{v}(0) = z',$$

where the process $\{\int_0^t \psi(s)\mathrm{d}s,\ t \in J\}$ has the same law as

$$\left\{W(t) - \int_0^t \mathsf{P}_N[B(u_s) - \lambda u_s]\mathrm{d}s,\ t \in J\right\}.$$

Let $\tilde{u}_t$ be a solution of

$$\dot{\tilde{u}} + B(\tilde{u}) + L\tilde{u} + \mathsf{P}_N[\lambda\tilde{u} - B(\tilde{u})] = h + \psi(t), \quad \tilde{u}(0) = z.$$

Then $\{\tilde{u}_t, t \in J\}$ has the same law as $\{u_t, t \in J\}$. Now the coupling operators $\mathcal{R}$ and $\mathcal{R}'$ are defined by

$$\mathcal{R}_t(z, z', \omega) = \tilde{u}_t, \quad \mathcal{R}'_t(z, z', \omega) = \tilde{u}'_t, \quad z, z' \in H, \, \omega \in \hat{\Omega}, \, t \in J.$$

By Proposition 3.2, for any $\varepsilon > 0$, $N \geq N_1$, and $\lambda \geq N^2/2$, we have

$$\begin{aligned} &\hat{\mathbb{P}}\{\exists t \in J \text{ s.t. } \tilde{v}_t \neq \tilde{u}'_t\} \\ &\quad \leq \varepsilon^a + 2\left[\exp\left(C_{\lambda,N}\varepsilon^{a-2}\|z - z'\|^2 e^{C(\|z\|^2 + \|z'\|^2)}\right) - 1\right]^{1/2}. \end{aligned} \tag{3.3}$$

Let $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k), k \geq 0$ be a sequence of independent copies of $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and $(\Omega, \mathcal{F}, \mathbb{P})$ the direct product of $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k)$. For any $\omega = (\omega^1, \omega^2, \dots) \in \Omega$ and $z, z' \in H$, we set $\tilde{u}_0 = z$, $\tilde{u}'_0 = z'$, and

$$\begin{aligned} &\tilde{u}_t(\omega) = \mathcal{R}_s(\tilde{u}_k(\omega), \tilde{u}'_k(\omega), \omega^k), \qquad \tilde{u}'_t(\omega) = \mathcal{R}'_s(\tilde{u}_k(\omega), \tilde{u}'_k(\omega), \omega^k), \\ &\tilde{v}_t(\omega) = \mathcal{Z}_s(\tilde{u}_k(\omega), \tilde{u}'_k(\omega), \omega^k), \end{aligned}$$

where $t = s + k, s \in [0, 1)$. We shall say that $(\tilde{u}_t, \tilde{u}'_t)$ is a *coupled trajectory at level* (N, λ) issued from (z, z').

3.2 Proof of Theorem 3.1

Step 1: Stratification. Let us take any $V, \psi \in \mathcal{V}$ and $z, z' \in X_R$ such that $d := \|z - z'\| \leq 1$. We need to prove the uniform equicontinuity of the family $\{g_t, t \geq 0\}$ on X_R, where

$$g_t = \|\mathfrak{P}_t^V \mathbf{1}\|_R^{-1} \mathfrak{P}_t^V \psi.$$

Without loss of generality, we can assume that ψ and V are non-negative, $\psi \leq 1$, and the integer N in representation (0.10) is the same for ψ and V (we denote it by N_0). Let $(u_t, u'_t) := (\tilde{u}_t, \tilde{u}'_t)$ be a coupled trajectory at level (N, λ) issued from (z, z') and let $v_t := \tilde{v}_t$ be the associated process. The parameters $N \geq N_0$ and $\lambda \geq N^2/2$ will be chosen later.

Following [22, 14], for any integers $r \geq 0$ and $\rho \geq 1$, we introduce the events [7]

$$\begin{aligned} &\bar{G}_r = \bigcap_{j=0}^{r} G_j, \quad G_j = \{v_t = u'_t, \forall t \in (j, j+1]\}, \quad F_{r,0} = \varnothing, \\ &F_{r,\rho} = \left\{ \sup_{t \in [0,r]} \left(\int_0^t \left(\|u_s\|_1^2 + \|u'_s\|_1^2\right) \mathrm{d}s - Kt \right) \leq \|z\|^2 + \|z'\|^2 + \rho; \right. \\ &\qquad\qquad \left. \|u_r\|^2 + \|u'_r\|^2 \leq \rho \right\}, \end{aligned}$$

[7] The event $\bar{G}_r$ is well defined also for $r = +\infty$.

where K is the constant in (5.19), and the pairwise disjoint events

$$A_0 = G_0^c,\ \ A_{r,\rho} = \big(\bar{G}_{r-1} \cap G_r^c \cap F_{r,\rho}\big) \setminus F_{r,\rho-1},\ r \geq 1,\ \rho \geq 1,\ \ \tilde{A} = \bar{G}_{+\infty}$$

We decompose as follows

$$\begin{aligned}
\mathfrak{P}_t^V \psi(z) - \mathfrak{P}_t^V \psi(z') &= \mathbb{E}\big\{\mathbb{I}_{A_0}\big[\Xi_t^V \psi(u_t) - \Xi_t^V \psi(u_t')\big]\big\} \\
&\quad + \sum_{r,\rho=1}^{\infty} \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}}\big[\Xi_t^V \psi(u_t) - \Xi_t^V \psi(u_t')\big]\big\} \\
&\quad + \mathbb{E}\big\{\mathbb{I}_{\tilde{A}}\big[\Xi_t^V \psi(u_t) - \Xi_t^V \psi(u_t')\big]\big\} \\
&= I_0^t + \sum_{r,\rho=1}^{\infty} I_{r,\rho}^t + \tilde{I}^t,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
I_0^t &= \mathbb{E}\big\{\mathbb{I}_{A_0}[\Xi_t^V \psi(u_t) - \Xi_t^V \psi(u_t')]\big\}, \\
I_{r,\rho}^t &= \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}}[\Xi_t^V \psi(u_t) - \Xi_t^V \psi(u_t')]\big\}, \\
\tilde{I}^t &= \mathbb{E}\big\{\mathbb{I}_{\tilde{A}}\big[\Xi_t^V \psi(u_t) - \Xi_t^V \psi(u_t')\big]\big\}.
\end{aligned}$$

In Steps 2 and 3, we estimate I_0^t, $I_{r,\rho}^t$, and $\tilde{I}^t$.

Step 2: Estimates for I_0^t and $I_{r,\rho}^t$. We have following inequalities

$$|I_0^t| \leq C_1(R,V)\|\mathfrak{P}_t^V \mathbf{1}\|_R\, \mathbb{P}(A_0)^{1/2}, \tag{3.5}$$

$$|I_{r,\rho}^t| \leq C_2(R,V) e^{r\|V\|_\infty} \|\mathfrak{P}_t^V \mathbf{1}\|_R\, \mathbb{P}(A_{r,\rho})^{1/2} \tag{3.6}$$

for any integers $r, \rho \geq 1$ and $R \geq R_0$, where R_0 is the number in Proposition 2.1. Let us prove (3.6), the other inequality is proved in a similar way. First assume that $r+1 \leq t$. Using $\psi \leq 1$, the positivity of $\Xi_t^V \psi$, and the Markov property, we derive

$$\begin{aligned}
I_{r,\rho}^t &\leq \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}} \Xi_t^V \psi(u_t)\big\} \leq \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}} \Xi_t^V\big\} \\
&= \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}} \mathbb{E}\big[\Xi_t^V \,\big|\, \mathcal{F}_{r+1}\big]\big\} \leq e^{r\|V\|_\infty} \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}} (\mathfrak{P}_{t-r-1}^V \mathbf{1})(u_{r+1})\big\},
\end{aligned}$$

where $\{\mathcal{F}_t\}$ is the filtration generated by (u_t, u_t'). Then from the positivity of V and (2.1) it follows that

$$\mathfrak{P}_{t-r-1}^V \mathbf{1}(y) \leq \mathfrak{P}_t^V \mathbf{1}(y) \leq M \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \mathfrak{w}(y), \quad y \in H,$$

so that

$$\begin{aligned}
I_{r,\rho}^t &\leq C_3 e^{r\|V\|_\infty} \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}} \mathfrak{w}(u_r)\big\} \\
&\leq C_3 e^{r\|V\|_\infty} \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \big\{\mathbb{P}(A_{r,\rho})\, \mathbb{E}\, \mathfrak{w}^2(u_r)\big\}^{1/2}.
\end{aligned}$$

Using this, (5.23), and the symmetry, we obtain (3.6). If $r > t$, then

$$I^t_{r,\rho} \le e^{r\|V\|_\infty}\mathbb{P}(A_{r,\rho}) \le e^{r\|V\|_\infty}\|\mathfrak{P}^V_t\mathbf{1}\|_R\,\mathbb{P}(A_{r,\rho})^{1/2},$$

which implies (3.6) by symmetry.

Step 3: Estimate for $\tilde I^t$. Let us show that

$$|\tilde I^t| \le C_4(R,V,\lambda,N,\psi)\|\mathfrak{P}^V_t\mathbf{1}\|_R d. \tag{3.7}$$

Indeed, we write

$$\tilde I^t = \mathbb{E}\{\mathbb{I}_{\tilde A}\Xi^V_t[\psi(u_t)-\psi(u'_t)]\} + \mathbb{E}\{\mathbb{I}_{\tilde A}[\Xi^V_t - \Xi^{V'}_t]\psi(u'_t)\} =: J^t_1 + J^t_2,$$

where $\Xi^{V'}_t := \exp\left(\int_0^t V(u'_s)\mathrm{d}s\right)$. Then by (5.3), on the event $\tilde A$ we have

$$\|\mathsf{P}_N(u_s-u'_s)\| \le e^{-\lambda s}d, \quad s\in[0,t].$$

Since $\psi \in L_b(H)$, we derive from this

$$|J^t_1| \le \mathbb{E}\{\mathbb{I}_{\tilde A}\Xi^V_t|\psi(u_t)-\psi(u'_t)|\} \le \|\psi\|_L e^{-\lambda t}\|\mathfrak{P}^V_t\mathbf{1}\|_R d \le \|\psi\|_L\|\mathfrak{P}^V_t\mathbf{1}\|_R d.$$

Similarly, as $V\in L_b(H)$,

$$\begin{aligned}|J^t_2| \le \mathbb{E}\{\mathbb{I}_{\tilde A}|\Xi^V_t - \Xi^{V'}_t|\} &\le \mathbb{E}\left\{\mathbb{I}_{\tilde A}\Xi^V_t\left[\exp\left(\int_0^t |V(u_s)-V(u'_s)|\mathrm{d}s\right)-1\right]\right\}\\ &\le \left[\exp\left(C_{\lambda,N}\lambda^{-1}\|V\|_L d(1-e^{-\lambda t})\right)-1\right]\|\mathfrak{P}^V_t\mathbf{1}\|_R\\ &\le \left[\exp\left(C_5(R,V,\lambda,N)d\right)-1\right]\|\mathfrak{P}^V_t\mathbf{1}\|_R.\end{aligned}$$

Recalling that $d\le 1$ and combining the estimates for J^t_1 and J^t_2, we get (3.7).

Step 4: Uniform equicontinuity of g_t. We use the following lemma, which is proved at the end of this subsection.

Lemma 3.3. *For any $\alpha>0$, there is an integer $N_2(\alpha)\ge 1$ and positive numbers a and β such that*

$$\mathbb{P}\{A_0\} \le C_6(R,\lambda,N)d^{a/2}, \tag{3.8}$$

$$\mathbb{P}\{A_{r,\rho}\} \le C_7(R)\left\{\left(d^a e^{-a\alpha r} + \left[\exp\left(C_8(R,\lambda,N)d^a e^{C'\rho - a\alpha r}\right)-1\right]^{1/2}\right)\wedge e^{-\beta\rho}\right\} \tag{3.9}$$

for any $N\ge N_2(\alpha)$, $\lambda\ge N^2/2$, $R\ge 1$, and a universal constant $C'>0$.

From (3.4)-(3.9) it follows that, for any $z,z'\in X_R$, $t\ge 0$, $R\ge R_0$, and $\alpha>0$, we have

$$\begin{aligned}\left|g_t(z)-g_t(z')\right| \le C_9(R,V,\lambda,N,\psi)\Bigg(&d^{a/4}+d\\ &+\sum_{r,\rho=1}^\infty e^{r\|V\|_\infty}\left\{\left(d^{a/2}e^{-a\alpha r/2} + \left[\exp\left(C_8 d^a e^{C'\rho-a\alpha r}\right)-1\right]^{1/4}\right)\wedge e^{-\beta\rho/2}\right\}\Bigg),\end{aligned}$$

provided that $N \geq N_0 \vee N_1 \vee N_2(\alpha)$ and $\lambda \geq N^2/2$. When $d = 0$, the series on the right-hand side vanishes. So to prove the uniform equicontinuity of $\{g_t\}$, it suffices to show that the series converges uniformly in $d \in [0,1]$. Since its terms are positive and monotone, it suffices to show the converge for $d = 1$:

$$\sum_{r,\rho=1}^{\infty} e^{r\|V\|_\infty} \left\{ \left(e^{-a\alpha r/2} + \left[\exp\left(C_8 e^{C'\rho - a\alpha r} \right) - 1 \right]^{1/4} \right) \wedge e^{-\beta\rho/2} \right\} < \infty. \tag{3.10}$$

To prove this, we will assume that α is sufficiently large. Let

$$S_1 = \{(r,\rho) \in \mathbb{N}^2 : \rho \leq a\alpha r/(2C')\}, \quad S_2 = \mathbb{N}^2 \setminus S_1.$$

Then taking $\alpha > 16\|V\|_\infty/a$, we see that

$$\sum_{(r,\rho)\in S_1} e^{r\|V\|_\infty} \left(e^{-a\alpha r/2} + \left[\exp\left(C_8 e^{C'\rho - a\alpha r} \right) - 1 \right]^{1/4} \right)$$
$$\leq C_{10}(R,N) \sum_{(r,\rho)\in S_1} e^{r\|V\|_\infty} e^{-a\alpha r/8} \leq C_{11}(R,N) \sum_{r=1}^{\infty} e^{-a\alpha r/16} < \infty.$$

Choosing $\alpha > 8C'\|V\|_\infty/(a\beta)$, we get

$$\sum_{(r,\rho)\in S_2} e^{r\|V\|_\infty} e^{-\beta\rho/2} \leq C_{12} \sum_{\rho=1}^{\infty} e^{-\beta\rho/4} < \infty.$$

These two inequalities show that (3.10) holds.

Proof of Lemma 3.3. Taking $\varepsilon = d$ in (3.3) and using $d \leq 1$, we get

$$\mathbb{P}\big\{A_0\big\} \leq d^a + 2\left[\exp\left(C_{\lambda,N} d^a e^{2CR^2} \right) - 1 \right]^{1/2} \leq C_6(R,\lambda,N) d^{a/2}$$

for $N \geq N_1$ and $\lambda \geq N^2/2$. This gives (3.8). From the inclusion $A_{r,\rho} \subset F^c_{r,\rho-1}$ and inequalities (5.19) and (5.21) it follows that

$$\mathbb{P}\{A_{r,\rho}\} \leq C_{13}(R) e^{-\beta\rho}, \tag{3.11}$$

where $\beta := \gamma_0/2$. By Proposition 5.1, on the event $A_{r,\rho}$ we have

$$\|u_r - u'_r\| \leq \exp(-\alpha r + c(\|z\|^2 + \|z'\|^2 + \rho)) d \leq C_{13}(R) e^{-\alpha r + c\rho} d, \tag{3.12}$$

provided that $N \geq N'_1(\alpha) := \sqrt{\alpha + cK}$ and $\lambda \geq N^2/2$. Recall that on the same event we have also

$$\|u_r\|^2 + \|u'_r\|^2 \leq \rho. \tag{3.13}$$

Using the Markov property, (3.3) with $\varepsilon = de^{-\alpha r}$, (3.12) and (3.13), we obtain

$$\mathbb{P}\{A_{r,\rho}\} \le \mathbb{P}\{\bar{G}_{r-1} \cap G_r^c \cap F_{r,\rho}\} = \mathbb{E}\{\mathbb{I}_{\bar{G}_{r-1}\cap F_{r,\rho}} \mathbb{E}(\mathbb{I}_{G_r^c} \mid \mathcal{F}_r)\} \le d^a e^{-a\alpha r}$$
$$+2\mathbb{E}\Big\{\mathbb{I}_{\bar{G}_{r-1}\cap F_{r,\rho}}\Big[\exp\Big(C_{\lambda,N} d^{a-2} e^{-(a-2)\alpha r} \|u_r - u_r'\|^2 e^{C(\|u_r\|+\|u_r'\|)}\Big) - 1\Big]^{1/2}\Big\}$$
$$\le d^a e^{-a\alpha r} + 2\left[\exp\Big(C_8(R,\lambda,N) d^a e^{C'\rho - a\alpha r}\Big) - 1\right]^{1/2}.$$

Combining this with (3.11) and taking $N \ge N_2(\alpha) := N_1 \vee N_1'(\alpha)$ and $\lambda \ge N^2/2$, we get the required inequality (3.9). $\square$

4 Proof of Theorem 1.1

The results of Sections 2 and 3 show that the conditions of Theorem 5.6 are satisfied if we choose

$$P_t^V(u,\Gamma) = (\mathfrak{P}_t^{V*}\delta_u)(\Gamma), \;\; X = H, \;\; X_R = B_{H^1}(R), \;\; R \ge R_0,$$
$$\mathfrak{w}(u) = \mathfrak{w}_m(u) = 1 + \|u\|^{2m}, \;\; \mathcal{C} = \mathcal{V}, \;\; V \in \mathcal{V}$$

with sufficiently large m and R_0. Thus there are eigenvectors $\mu_V \in \mathcal{P}(H)$ and $h_V \in L^\infty_{\mathfrak{w}}(H)$ corresponding to an eigenvalue $\lambda_V > 0$. Moreover, for any $R \ge 1$, the restriction of h_V to X_R is continuous and strictly positive, so $h_V : H^1 \to \mathbb{R}$ is continuous and strictly positive. As $\mathbb{P}_u\{u_1 \in H^1\} = 1$ and $h(u) = \lambda_V^{-1}\mathfrak{P}_1^V h_V(u)$, we have

$$h_V(u) \ge \lambda_V^{-1} e^{-\|V\|_\infty} \mathbb{E}_u h_V(u_1) > 0 \quad u \in H.$$

The continuity of $h_V : H \to \mathbb{R}$ follows from the uniform convergence in (1.4), and the uniqueness of μ_V and h_V from (1.4) and (1.5). The proof of (1.4) is carried out in Steps 1-3, and that of (1.6) in Step 4. Convergence (1.5) follows immediately from (1.4).

Step 1: Proof of (1.4) *for* $f \in \mathcal{V}$. In view of (5.32), for any $f \in \mathcal{V}$, we have limit (1.4) in $C(X_R) \cap L^1(H,\mu_V)$. We claim that this limit holds also in $C(B_H(R))$ for any $R \ge 1$. Indeed, it suffices to check condition (5.33) with $B = B_H(R)$ and $s = 1$, i.e.,

$$A_{R,r} := \sup_{u\in B_H(R)} \int_{H\setminus X_r} \mathfrak{w}_m(v)\, P_1^V(u,\mathrm{d}v) \to 0 \quad \text{as } r \to \infty.$$

From the Poincaré inequality and (5.24) it follows that

$$\begin{aligned} A_{R,r} &\le r^{-2} \sup_{u\in B_H(R)} \mathbb{E}_u\left\{\mathfrak{w}_m(u_1)\|u_1\|_1^2\, \Xi_1^V\right\} \\ &\le r^{-2} e^{\|V\|_\infty} \sup_{u\in B_H(R)} \mathbb{E}_u\left\{(1+\alpha_1^{-m}\|u_1\|_1^{2m})\|u_1\|_1^2\right\} \\ &\le r^{-2} C(m) R^{8(m+1)} \to 0 \quad \text{as } r \to \infty. \end{aligned}$$

This implies (1.4) for $f \in \mathcal{V}$.

Step 2: Proof of (1.4) *for* $f \in C_b(H)$. For any $n \geq 1$, let $\tilde{f}_n \in L_b(H)$ be such that

$$\sup_{u \in B_H(n)} |\tilde{f}_n(u) - f(u)| \leq \frac{1}{n}.$$

Then the functions $f_n = \tilde{f}_n \circ \mathsf{P}_n$ belong to the space $\mathcal{V}$, satisfy $\|f_n\|_\infty \leq \|f\|_\infty$ and $f_n \to f$ as $n \to \infty$, uniformly on compact subsets of H. Setting

$$\Delta_t(g) = \sup_{u \in B_H(R)} \left|\lambda_V^{-t}\mathfrak{P}_t^V g(u) - \langle g, \mu_V\rangle h_V(u)\right|, \quad \|g\|_{0,R} = \sup_{u \in B_H(R)} |g(u)|,$$

for any $t \geq 0$ and $n \geq 1$, we write

$$\Delta_t(f) \leq \Delta_t(f_n) + \|h_V\|_{0,R}\, |\langle f - f_n, \mu_V\rangle| + \lambda_V^{-t}\|\mathfrak{P}_t^V(f - f_n)\|_{0,R}.$$

Since $f_n \in \mathcal{V}$, the first term on the right-hand side of this inequality goes to zero as $k \to \infty$ for any fixed $n \geq 1$. The Lebesgue theorem on dominated convergence implies that $|\langle f - f_n, \mu_V\rangle| \to 0$ as $n \to \infty$. Thus, the required convergence will be established if we show that

$$\sup_{t \geq 1} \lambda_V^{-t}\|\mathfrak{P}_t^V(f - f_n)\|_{0,R} \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

To prove this limit, we take any $\rho > 0$ and write

$$\|\mathfrak{P}_t^V(f - f_n)\|_{0,R} \leq J_1(t, n, \rho) + J_2(t, n, \rho), \tag{4.2}$$

where

$$J_1(t, n, \rho) = \|\mathfrak{P}_t^V\big((f - f_n)\mathbb{I}_{X_\rho}\big)\|_{0,R}, \quad J_2(t, n, \rho) = \|\mathfrak{P}_t^V\big((f - f_n)\mathbb{I}_{X_\rho^c}\big)\|_{0,R}.$$

By (2.2), we have

$$J_1(t, n, \rho) \leq \varepsilon(n, \rho)\, \|\mathfrak{P}_t^V \mathbf{1}\|_{0,R} \leq \varepsilon(n, \rho)\, C_R\, \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0},$$

where $\varepsilon(n, \rho) = \|f - f_n\|_{X_\rho} \to 0$ as $n \to \infty$. Convergence (1.4) with $f = \mathbf{1}$ implies that

$$\text{the set } \{\lambda_V^{-t}\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}\}_{t \geq 0} \text{ is bounded in } \mathbb{R}. \tag{4.3}$$

It follows that

$$\sup_{t \geq 0} \lambda_V^{-t} J_1(t, n, \rho) \to 0 \quad \text{as } n \to \infty. \tag{4.4}$$

To estimate J_2, we use (2.8). For any $\rho, n \geq 1$ and $t \geq 0$, we have

$$\begin{aligned} \lambda_V^{-t} J_2(t, n, \rho) &\leq 2\|f\|_\infty \rho^{-2} \lambda_V^{-t}\|\mathfrak{P}_t^V F\|_{0,R} \\ &\leq C_R \|f\|_\infty \rho^{-2} \lambda_V^{-t}\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}. \end{aligned}$$

By (4.3), the right-hand side of this inequality goes to zero as $\rho \to \infty$, uniformly with respect to $t \geq 1$. Combining this with (4.4), we see that supremum over

$t \geq 1$ of the right-hand side of (4.2) can be made arbitrarily small by choosing first $\rho > 0$ and then $n \geq 1$ sufficiently large. This proves (4.1).

Step 3: Proof of (1.4) *for* $f \in C_{\mathfrak{m}}(H)$. We use again an approximation argument. Let us fix any $\varkappa \in (0, \gamma_0)$ and $f \in C_{\mathfrak{m}}(H)$ with $\mathfrak{m} = \mathfrak{m}_\varkappa$. We define a sequence $\{f_n\}$ by the relation $f_n = f^+ \wedge n - f^- \wedge n$. Then $f_n \in C_b(H)$, $|f_n| \leq |f|$ for any $n \geq 1$, and $f_n \to f$ in $L^\infty_{\mathfrak{m}'}(H)$ with $\mathfrak{m}' = \mathfrak{m}_{\varkappa'}$ for any $\varkappa' \in (\varkappa, \gamma_0)$. Furthermore, in view of (1.4) and the Lebesgue theorem on dominated convergence, we have

$$\sup_{u \in B_H(R)} \left|\lambda_V^{-t} \mathfrak{P}_t^V f_n(u) - \langle f_n, \mu_V \rangle h_V(u)\right| \to 0 \quad \text{as } t \to \infty \text{ for any fixed } n \geq 1,$$

$$|\langle f - f_n, \mu_V \rangle| \to 0 \quad \text{as } n \to \infty.$$

Thus, as in the previous step, it suffices to prove that

$$\sup_{t \geq 0} \lambda_V^{-t} \|\mathfrak{P}_t^V (f - f_n)\|_{0,R} \to 0 \quad \text{as } n \to \infty \tag{4.5}$$

To see this, we use (2.7) for $\mathfrak{m}'$:

$$\|\mathfrak{P}_t^V (f - f_n)\|_{0,R} \leq \varepsilon_n \|\mathfrak{P}_t^V \mathfrak{m}'\|_{0,R} \leq C_R \, \varepsilon_n \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0},$$

where $\varepsilon_n = \|f - f_n\|_{L^\infty_{\mathfrak{w}'}} \to 0$ as $n \to \infty$. Combining this with (4.3), we get (4.5).

Step 4: Proof of (1.6). In view of (1.4), it suffices to show that

$$\sup_{(t,\nu) \in \mathbb{R}_+ \times \Lambda(\varkappa', M)} \left\{ \int_{B_H(R)^c} \left|\lambda_V^{-t} \mathfrak{P}_t^V f - \langle f, \mu_V \rangle h_V\right| \nu(\mathrm{d}u) \right\} \to 0 \text{ as } R \to \infty. \tag{4.6}$$

By (2.7) and (4.3), we have

$$\|\mathfrak{P}_t^V f\|_{L^\infty_{\mathfrak{m}}} \leq C_1 \|\mathfrak{P}_t^V \mathbf{1}\|_{R_0} \leq C_2 \lambda_V^t \quad \text{for all } t \geq 0.$$

It follows that

$$\left|\lambda_V^{-k} \mathfrak{P}_t^V f(u)\right| \leq C_3 \mathfrak{m}_\varkappa(u).$$

Since $\varkappa < \varkappa'$, $h_V \in C_{\mathfrak{w}}(H)$, and

$$\sup_{\nu \in \Lambda(\varkappa', M)} \int_{B_H(R)^c} \mathfrak{m}_\varkappa(u) \, \nu(\mathrm{d}u) \leq M e^{(\varkappa - \varkappa')R^2} \to 0 \quad \text{as } R \to \infty,$$

we obtain (4.6). This completes the proof of Theorem 1.1.

5 Appendix

5.1 The Foiaş–Prodi estimate

Let us take any numbers $T, \lambda > 0$, any function $\varphi \in L^2([0,T]; H)$, any integer $N \geq 1$, and consider the equations

$$\dot{u} + B(u) + Lu = h(x) + \partial_t \varphi(t,x), \tag{5.1}$$

$$\dot{v} + B(v) + Lv + \mathsf{P}_N[\lambda(v - u) + B(u) - B(v)] = h(x) + \partial_t \varphi(t,x), \tag{5.2}$$

where P_N is the orthogonal projection in H onto the space H_N defined by (0.11). The following result is a version of the Foiaş–Prodi estimate obtained in [8]; see also Section 2.1.8 in [20] for a similar result for the Navier–Stokes system (with different equation instead of (5.2)) and Section 7.3 in [22] for the damped nonlinear wave equation.

Proposition 5.1. *Let $u, v \in C([0,T];H) \cap L^2([0,T];H^1)$ be solutions of (5.1) and (5.2) issued from z and z', respectively. Then*

$$\|\mathsf{P}_N(u_t - v_t)\| \le e^{-\lambda t}\|\mathsf{P}_N(z - z')\|, \quad t \in [0,T]. \tag{5.3}$$

If we assume additionally that

$$\int_0^t (\|u_s\|_1^2 + \|v_s\|_1^2)\,\mathrm{d}s \le \rho + Kt, \quad t \in [0,T] \tag{5.4}$$

for some numbers $\rho > 0$ and $K > 0$, then for any $\alpha > 0$, we have

$$\|u_t - v_t\| \le C_{\lambda,N} e^{-\alpha t + c\rho}\|z - z'\|, \quad t \in [0,T], \tag{5.5}$$

provided that $2\lambda > N^2 \ge \alpha + cK$. Here $c > 0$ is an absolute constant and $C_{\lambda,N}$ is a constant depending on λ and N.

Proof. Step 1: Proof of (5.3). Let us set $y = \mathsf{P}_N(u - v)$. Then

$$\dot{y} + Ly + \lambda y = 0.$$

Taking the scalar product in H of this equation with y, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2 + \|y\|_1^2 + \lambda\|y\|^2 = 0.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2 + 2\lambda\|y\|^2 \le 0,$$

which implies (5.3).

Step 2: Proof of (5.5). Let $w = u - v$. Then

$$\dot{w} + Lw + \lambda\mathsf{P}_N w + \mathsf{Q}_N[B(u) - B(v)] = 0, \tag{5.6}$$

where $\mathsf{Q}_N = 1 - \mathsf{P}_N$. For any $a, b \in H^1$, let us set $B(a,b) = \Pi(\langle a, \nabla\rangle b)$. Taking the scalar product of (5.6) with w, and using the equality

$$B(v) - B(u) = B(v,w) + B(w,u), \tag{5.7}$$

we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \|w\|_1^2 + \lambda\|\mathsf{P}_N w\|^2 &= \langle B(v) - B(u), \mathsf{Q}_N w\rangle \\ &= \langle B(v,w), \mathsf{Q}_N w\rangle + \langle B(w,u), \mathsf{Q}_N w\rangle \\ &=: I_1 + I_2. \end{aligned} \tag{5.8}$$

Using the identity

$$\langle B(a,b),b\rangle = 0, \quad a,b\in H^1$$

and the Hölder inequality, we obtain

$$\begin{aligned}|I_1| = |\langle B(v,\mathsf{P}_N w),\mathsf{Q}_N w\rangle| &\le C_1\int_{\mathbb{T}^2}|v||\nabla\mathsf{P}_N w||\mathsf{Q}_N w|\mathrm{d}x\\ &\le C_1\|v\|\|\nabla\mathsf{P}_N w\|_\infty\|w\| \le \frac{1}{2}\|\nabla\mathsf{P}_N w\|_\infty^2 + C_2\|v\|^2\|w\|^2. \end{aligned} \tag{5.9}$$

To estimate I_2, we use the Hölder inequality, the inclusion $H^{\frac{1}{2}}\subset L^4$, and the interpolation inequality $\|a\|_{1/2}^2\le\|a\|\|a\|_1$:

$$\begin{aligned}|I_2| = |\langle B(w,u),\mathsf{Q}_N w\rangle| &\le C_3\int_{\mathbb{T}^2}|w||\nabla u||\mathsf{Q}_N w|\mathrm{d}x\\ &\le C_3\|w\|_{L^4}\|u\|_1\|\mathsf{Q}_N w\|_{L^4}\le C_4\|w\|\|w\|_1\|u\|_1\le\frac{1}{2}\|w\|_1^2+C_5\|w\|^2\|u\|_1^2.\end{aligned}$$

Combining this with (5.8) and (5.9), and using the Poincaré inequality

$$N\|\mathsf{Q}_N w\|\le\|w\|_1,$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2+\left(l-c_1(\|u\|_1^2+\|v\|_1^2)\right)\|w\|^2\le\|\nabla\mathsf{P}_N w\|_\infty^2, \tag{5.10}$$

where $l=\min\{N^2,2\lambda\}$. From (5.3) we deduce that

$$\|\nabla\mathsf{P}_N w_t\|_\infty^2\le C_N e^{-2\lambda t}\|w_0\|^2.$$

Hence, (5.10) and (5.4) imply that

$$\|w_t\|^2\le\left(1+C_N\int_0^t e^{(l-2\lambda)s}\mathrm{d}s\right)\|w_0\|^2\exp\left(-lt+c_1(\rho+Kt)\right).$$

Choosing λ and N such that $2\lambda>N^2\ge 2\alpha+c_1K$, we get (5.5) with $c=c_1/2$. $\square$

5.2 Proof of Proposition 3.2

We closely follow the arguments of the proof of a similar result from Section 7.3 of [22] in the case of the nonlinear wave equation (see also Section 3.3.3 of [20]).

Note that inequality (3.2) concerns the laws of the solutions and not the solutions themselves. Thus we can choose the underlying probability space $(\Omega,\mathcal{F},\mathbb{P})$. We assume that Ω is the space $C(\mathbb{R}_+;\mathbb{R})$ endowed with the topology of uniform convergence on bounded intervals, $\mathbb{P}$ is the law of the Wiener process W in (0.3), and $\mathcal{F}$ is the completion of the Borel σ-algebra of Ω with respect to $\mathbb{P}$. We define a stopping time by

$$\tau^u=\inf\{t\ge 0:\mathcal{E}^u(t)\ge\|z\|^2+Kt+\rho\},$$

where $\mathcal{E}^u(t)$ is the functional and K is the number in Lemma 5.3, and $\rho > 0$ is a constant to be chosen later. The stopping times $\tau^{u'}$ and τ^v are defined in a similar way. Then by inequality (5.19), we have

$$\mathbb{P}\{\tau^u < \infty\} + \mathbb{P}\{\tau^{u'} < \infty\} \leq 2e^{-\gamma_0 \rho}. \tag{5.11}$$

We define a transformation $\Lambda : \Omega \to \Omega$ by

$$\Lambda(\omega)(t) = \omega(t) - \int_0^t \varphi(s, \omega) \mathrm{d}s, \quad \varphi(t, \omega) = \mathbb{I}_{\{t \leq \tilde{\tau}\}} \mathsf{P}_N[\lambda(v - u) + B(u) - B(v)],$$

where $\tilde{\tau} = \tau^u \wedge \tau^{u'} \wedge \tau^v$ and $\mathbb{I}_{\{t \leq \tilde{\tau}\}}$ is the indicator function of the interval $[0, \tilde{\tau}]$. We use the following result, whose proof is given at the end of this section.

Lemma 5.2. *There is an integer $N_1 \geq 1$ such that for any numbers $N \geq N_1$, $\lambda \geq N^2/2$, and $\rho > 0$ and any initial points $z, z' \in H$, we have*

$$\|\Lambda_* \mathbb{P} - \mathbb{P}\|_{var} \leq \left[\exp\left(C_{\lambda,N} \|z - z'\|^2 e^{C(\|z\|^2 + \|z'\|^2 + \rho)}\right) - 1\right]^{1/2}, \tag{5.12}$$

where $\Lambda_* \mathbb{P}$ stands for the image of $\mathbb{P}$ under Λ, and C and $C_{\lambda,N}$ are positive constants not depending on ρ, z, z'.

Let us introduce auxiliary processes $y_{u'}$ and y_v in H defined as follows: for $t \leq \tilde{\tau}$ they coincide with the processes u' and v, respectively, while for $t \geq \tilde{\tau}$ and $\tilde{\tau} < \infty$ they are zero. With probability 1, we have

$$y_v(t, \omega) = y_{u'}(t, \Lambda(\omega)), \quad t \in J. \tag{5.13}$$

Let us denote by u_1' and v_1 the restrictions of $u'(t)$ and $v(t)$ to J. Then

$$\begin{aligned}\|\nu(z, z') - \nu'(z')\|_{var} &= \sup_\Gamma |\mathbb{P}\{v_1 \in \Gamma\} - \mathbb{P}\{u_1' \in \Gamma\}| \\ &\leq \mathbb{P}\{\tilde{\tau} < \infty\} + \sup_\Gamma |\mathbb{P}\{v_1 \in \Gamma, \tilde{\tau} = \infty\} - \mathbb{P}\{u_1' \in \Gamma, \tilde{\tau} = \infty\}| = \mathcal{L}_1 + \mathcal{L}_2,\end{aligned}$$

where the supremum is taken over all Borel subsets of $C(J; H)$. Note that

$$\mathcal{L}_2 \leq \|\Lambda_* \mathbb{P} - \mathbb{P}\|_{var}.$$

Further, we have

$$\mathcal{L}_1 \leq \mathbb{P}\{\tau^v < \infty, \tau^u \wedge \tau^{u'} = \infty\} + \mathbb{P}\{\tau^u < \infty\} + \mathbb{P}\{\tau^{u'} < \infty\}.$$

Moreover, thanks to (5.13),

$$\begin{aligned}\mathbb{P}\{\tau^v < \infty, \tau^u \wedge \tau^{u'} = \infty\} &\leq \mathbb{P}\{\tau^{y_v} < \infty\} = \Lambda_* \mathbb{P}\{\tau^{y_{u'}} < \infty\} \\ &\leq \mathbb{P}\{\tau^{y_{u'}} < \infty\} + \|\Lambda_* \mathbb{P} - \mathbb{P}\|_{var} \\ &\leq \mathbb{P}\{\tau^{u'} < \infty\} + \|\Lambda_* \mathbb{P} - \mathbb{P}\|_{var}.\end{aligned}$$

Combining last four inequalities, we infer that

$$\|\nu(z,z')-\nu'(z')\|_{var} \le 2\left(\mathbb{P}\{\tau^u<\infty\}+\mathbb{P}\{\tau^{u'}<\infty\}+\|\Lambda_*\mathbb{P}-\mathbb{P}\|_{var}\right).$$

Finally using this with inequalities (5.11) and (5.12), we get

$$\|\nu(z,z')-\nu'(z')\|_{var} \le 4\,e^{-\gamma_0\rho}+2\left[\exp\left(C_{\lambda,N}\|z-z'\|^2 e^{C(\|z\|^2+\|z'\|^2+\rho)}\right)-1\right]^{1/2}.$$

Choosing $a=2\gamma_0/(\gamma_0+1)$ and $\rho=-\gamma_0^{-1}a\ln(\varepsilon/4^{1/a})$, we obtain (3.2).

Proof of Lemma 5.2. Step 1: Girsanov theorem. We write $\Omega=\Omega_N\dot{+}\Omega_N^\perp$, where $\Omega_N=C(\mathbb{R}_+;H_N)$ and $\Omega_N^\perp=C(\mathbb{R}_+;H_N^\perp)$. For any $\omega=\omega_1\dot{+}\omega_2\in\Omega$, we write $\omega=(\omega_1,\omega_2)\in\Omega_N\times\Omega_N^\perp$. Then the transformation Λ can be written as $\Lambda(\omega)=(\Upsilon(\omega),\omega_2)$, where $\Upsilon:\Omega\to\Omega_N$ is given by

$$\Upsilon(\omega)(t)=\omega_1(t)+\int_0^t\varphi(s,\omega)\mathrm{d}s.$$

It is not difficult to see that

$$\|\Lambda_*\mathbb{P}-\mathbb{P}\|_{var}\le\int_{\Omega_N^\perp}\|\Upsilon_*(\mathbb{P}_N,\omega_2)-\mathbb{P}_N\|_{var}\mathbb{P}_N^\perp(\mathrm{d}\omega_2),$$

where $\mathbb{P}_N$ and $\mathbb{P}_N^\perp$ are the images of $\mathbb{P}$ under the projections $\hat{\mathsf{P}}_N:\Omega\to\Omega_N$ and $\hat{\mathsf{Q}}_N:\Omega\to\Omega_N^\perp$, respectively. Let

$$X=\omega_1(t),\quad \hat{X}=\omega_1(t)+\int_0^t\varphi(s,\omega)\mathrm{d}s.$$

Then $\mathbb{P}_N$ coincides with the law $\mathcal{D}(X)$ of the random variable X and $\Upsilon_*(\mathbb{P}_N,\omega_2)$ coincides with that of $\hat{X}$. By the Girsanov theorem (see Theorem A.10.1 in [20]), we have

$$\|\mathcal{D}(\hat{X})-\mathcal{D}(X)\|_{var}\le\frac{1}{2}\left(\left(\mathbb{E}\exp\left[6\max_{1\le j\le N}b_j^{-1}\int_0^\infty\|\varphi(t)\|^2\mathrm{d}t\right]\right)^{\frac{1}{2}}-1\right)^{\frac{1}{2}},\tag{5.14}$$

provided that the Novikov condition

$$\mathbb{E}\exp\left(p\int_0^\infty\|\varphi(t)\|^2\mathrm{d}t\right)<\infty\quad\text{ for any } p>0$$

is satisfied. In Step 2, we show that

$$\mathbb{E}\exp\left(p\int_0^\infty\|\varphi(t)\|^2\mathrm{d}t\right)\le\exp\left(C_{p,\lambda,N}\|z-z'\|^2e^{C(\|z\|^2+\|z'\|^2+\rho)}\right)\tag{5.15}$$

for any $p>0$. Clearly, this and (5.14) imply (5.12).

Step 2: Proof of (5.15). By Proposition 5.1, the following inequalities hold

$$\|\mathsf{P}_N(u_t - v_t)\| \le e^{-\lambda t}\|\mathsf{P}_N(z - z')\|, \quad t \ge 0, \tag{5.16}$$

$$\|u_t - v_t\| \le C_1 e^{-t + c(\|z\|^2 + \|z'\|^2 + \rho)}\|z - z'\|, \quad t \in [0, \tilde{\tau}], \tag{5.17}$$

if $2\lambda > N^2 \ge 1 + cK$. We have

$$\begin{aligned}
\mathbb{E}\exp\left(p\int_0^\infty \|\varphi(t)\|^2 \mathrm{d}t\right) &= \mathbb{E}\exp\left(p\int_0^{\tilde{\tau}} \|\varphi(t)\|^2 \mathrm{d}t\right) \\
&\le \mathbb{E}\exp\left(C_2\int_0^{\tilde{\tau}} \left(\|\mathsf{P}_N[u - v]\|^2 + \|\mathsf{P}_N[B(u) - B(v)]\|^2\right)\mathrm{d}t\right).
\end{aligned} \tag{5.18}$$

Integrating by parts and using the Hölder inequality, we see that

$$|\langle B(a, b), e_j\rangle| \le C'_j\|a\|_1\|b\|, \quad a, b \in H^1,\ j \ge 1.$$

Combining this with (5.7) and (5.16)-(5.18), we get

$$\begin{aligned}
&\mathbb{E}\exp\left(p\int_0^\infty \|\varphi(t)\|^2 \mathrm{d}t\right) \\
&\le \mathbb{E}\exp\left(C_3\|z - z'\|^2 \int_0^\infty e^{-t + c(\|z\|^2 + \|z'\|^2 + \rho)}(1 + \|z\|^2 + \|z'\|^2 + Kt + \rho)^2 \mathrm{d}t\right) \\
&\le \mathbb{E}\exp\left(C_4\|z - z'\|^2 \int_0^\infty e^{-t/2 + 2c(\|z\|^2 + \|z'\|^2 + \rho)}\mathrm{d}t\right) \\
&= \exp\left(2C_4\|z - z'\|^2 e^{C(\|z\|^2 + \|z'\|^2 + \rho)}\right).
\end{aligned}$$

This proves (5.15).

□

5.3 A priori estimates

The following lemma gathers some standard a priori estimates for the solutions of the stochastic Navier–Stokes system. The reader is referred to Section 2.4.2 in [20] for more general results.

Lemma 5.3. *Assume*[8] *that* $\mathfrak{B}_1 < \infty$, $h \in H^1$, *and* u_t *is a solution of* (0.5) *issued from* $u \in H$. *Then we have the following estimates.*

Exponential moments. *There are numbers* $\gamma_0 = \gamma_0(\mathfrak{B}_0) > 0$ *and* $K = K(\mathfrak{B}_0, \|h\|) > 0$ *such that for any* $\varkappa \in (0, \gamma_0)$,

$$\mathbb{P}_u\left\{\sup_{t\ge 0}\left(\mathcal{E}(t) - Kt\right) \ge \|u\|^2 + \rho\right\} \le e^{-\gamma_0\rho}, \quad \rho \ge 0, \tag{5.19}$$

$$\mathbb{E}_u e^{\varkappa\mathcal{E}(t)} \le C_1(\varkappa, \mathfrak{B}_0)e^{\varkappa(Kt + \|u\|^2)}, \tag{5.20}$$

$$\mathbb{E}_u \exp(\varkappa\|u_t\|^2) \le e^{-\varkappa t}\exp(\varkappa\|u\|^2) + C_2(\varkappa, \mathfrak{B}_0, \|h\|), \tag{5.21}$$

$$\mathbb{E}_u\left\{\|u_t\|^2\exp(\varkappa\|u_t\|^2)\right\} \le C_3(t, \varkappa, \mathfrak{B}_0, \|h\|)\exp(\varkappa\|u\|^2), \tag{5.22}$$

[8] Recall that $\mathfrak{B}_i = \sum_{j\ge 1}\alpha_j^i b_j^2$, $i = 0, 1$.

where $\mathcal{E}(t) = \mathcal{E}^u(t) := \|u_t\|^2 + \int_0^t \|u_s\|_1^2 \mathrm{d}s$.

Polynomial moments. *For any* $m \geq 1$,

$$\mathbb{E}_u \|u_t\|^{2m} \leq e^{-m\alpha_1 t}\|u\|^{2m} + C_4(m, \mathfrak{B}_0, \|h\|), \tag{5.23}$$
$$\mathbb{E}_u \|u_t\|_1^{2m} \leq C_5(t, m, \mathfrak{B}_1, \|h\|_1)\|u\|^{8m}. \tag{5.24}$$

Proof. Estimate (5.19) is established in Proposition 2.4.10 in [20]. To prove (5.20), we denote $\tilde{\mathcal{C}}_\rho := \mathcal{C}_{\rho-1} \setminus \mathcal{C}_\rho$, where $\mathcal{C}_\rho$ the event on the left-hand side of (5.19) and $\mathcal{C}_{-1} := \Omega$. Then for any $\varkappa \in (0, \gamma_0)$, we have

$$\begin{aligned}
\mathbb{E}_u e^{\varkappa \mathcal{E}(t)} &= \sum_{\rho=0}^{\infty} \mathbb{E}_u \left\{ e^{\varkappa \mathcal{E}(t)} I_{\tilde{\mathcal{C}}_\rho} \right\} \leq e^{\varkappa(Kt + \|u\|^2)} \sum_{\rho=0}^{\infty} e^{\varkappa \rho} \mathbb{P}\{\mathcal{C}_{\rho-1}\} \\
&\leq e^{\varkappa(Kt+\|u\|^2)+\gamma_0} \sum_{\rho=0}^{\infty} e^{(\varkappa-\gamma_0)\rho} = \frac{e^{\varkappa(Kt+\|u\|^2)+\gamma_0}}{1 - e^{(\varkappa-\gamma_0)}}.
\end{aligned}$$

Estimates (5.21) and (5.24) are proved in Propositions 2.4.9 and 2.4.12 in [20], respectively. To show [9] (5.23), we set $F(u) = \|u\|^{2m}$. Then

$$\begin{aligned}
&\partial_u F(u; v) = 2m\|u\|^{2(m-1)}\langle u, v\rangle, \\
&\partial_u^2 F(u; v) = 2m\|u\|^{2(m-1)}\|v\|^2 + 4m(m-1)\|u\|^{2(m-2)}\langle u, v\rangle^2,
\end{aligned}$$

so applying the Itô formula for the functional F and taking the expectation:

$$\begin{aligned}
\mathbb{E}_u \|u_t\|^{2m} = \|u\|^{2m} + \mathbb{E}_u \int_0^t \Big(&2m\|u_s\|^{2(m-1)}\langle u_s, -Lu_s - B(u_s) + h\rangle \\
&+ m\|u_s\|^{2(m-1)}\mathfrak{B}_0 + 2m(m-1)\|u_s\|^{2(m-2)} \sum_{j=1}^{\infty} b_j^2 u_j^2 \Big) \mathrm{d}s,
\end{aligned}$$

where $u_j = \langle u, e_j \rangle$. The identity

$$\langle u, B(u)\rangle = 0 \tag{5.25}$$

and the Cauchy–Schwarz and Poincaré inequalities imply that

$$\begin{aligned}
\mathbb{E}_u \|u_t\|^{2m} \leq \|u\|^{2m} + \mathbb{E}_u \int_0^t \Big(&2m\|u_s\|^{2(m-1)}(-\|u_s\|_1^2 + \|u_s\|\|h\|) \\
&+ m\|u_s\|^{2(m-1)}\mathfrak{B}_0 + 2m(m-1)\|u_s\|^{2(m-1)}\mathfrak{B}_0 \Big) \mathrm{d}s \\
\leq \|u\|^{2m} - m\alpha_1 \int_0^t &\mathbb{E}_u \|u_s\|^{2m} \mathrm{d}s + tC_6(m, \mathfrak{B}_0, \|h\|).
\end{aligned}$$

Combining this with the Gronwall inequality, we obtain (5.23).

[9] We confine ourselves to a formal derivation of (5.23). The accurate proof is based on the same arguments applied to the stopped solutions $u(t \wedge \tau_n)$, where $\tau_n = \inf\{t \geq 0 : \|u(t)\| > n\}$.

To prove (5.22), we apply the Itô formula for $F(t,u)=t\|u\|^2\exp(\varkappa\|u\|^2)$, use the equalities

$$\begin{aligned}
&\partial_t F(t,u;v)=\|u\|^2\exp(\varkappa\|u\|^2),\\
&\partial_u F(t,u;v)=2t\exp(\varkappa\|u\|^2)(1+\varkappa\|u\|^2)\langle u,v\rangle,\\
&\partial_u^2 F(t,u;v)=2t\exp(\varkappa\|u\|^2)\big(2\varkappa(2+\varkappa\|u\|^2)\langle u,v\rangle^2+(1+\varkappa\|u\|^2)\|v\|^2\big),
\end{aligned}$$

and take the expectation:

$$\begin{aligned}
t\mathbb{E}_u\left\{\|u_t\|^2\exp(\varkappa\|u_t\|^2)\right\}=\mathbb{E}_u\int_0^t\Bigg(&\|u_s\|^2+2s(1+\varkappa\|u_s\|^2)\langle u_s,-Lu_s-B(u_s)+h\rangle\\
&+s\sum_{j=1}^{\infty}\Big[2\varkappa(2+\varkappa\|u_s\|^2)b_j^2u_j^2+(1+\varkappa\|u_s\|^2)b_j^2\Big]\Bigg)\exp(\varkappa\|u_s\|^2)\mathrm{d}s.
\end{aligned}$$

Again using (5.25) and the Cauchy–Schwarz and Poincaré inequalities, we get for sufficiently small $\gamma_0=\gamma_0(\mathfrak{B}_0)>0$ and any $\varkappa\in(0,\gamma_0)$,

$$\begin{aligned}
t\mathbb{E}_u\left\{\|u_t\|^2\exp(\varkappa\|u_t\|^2)\right\}\le\mathbb{E}_u\int_0^t\Bigg(&\|u_s\|^2+2s(1+\varkappa\|u_s\|^2)(-\|u_s\|_1^2+\|u_s\|\|h\|)\\
&+s2\varkappa(2+\varkappa\|u_s\|^2)\|u_s\|^2\mathfrak{B}_0+s(1+\varkappa\|u_s\|^2)\mathfrak{B}_0\Bigg)\exp(\varkappa\|u_s\|^2)\mathrm{d}s\\
\le\mathbb{E}_u\int_0^t&\Big(\|u_s\|^2+sC_7(\varkappa,\mathfrak{B}_0,\|h\|)\Big)\exp(\varkappa\|u_s\|^2)\mathrm{d}s.
\end{aligned}$$

Thus (5.22) follows from (5.21), the Poincaré inequality, and the estimate

$$\mathbb{E}_u\left\{\int_0^t\|u_s\|_1^2\exp(\varkappa\|u_s\|^2)\mathrm{d}s\right\}\le C_8(t,\varkappa,\mathfrak{B}_0,\|h\|)\exp(\varkappa\|u\|^2).$$

The latter is easily proved by applying the Itô formula for $F(u)=\exp(\varkappa\|u\|^2)$. This completes the proof of the lemma.

□

5.4 Hyper-exponential recurrence

For any $R>0$, let $\tau(R)$ be the first hitting time of the set X_R:

$$\tau(R)=\inf\{t\ge 0:u_t\in X_R\}. \tag{5.26}$$

We have the following standard estimate for the exponential moment of $\tau(R)$.

Proposition 5.4. *For any $\gamma>0$, there are positive numbers m,R, and C such that*

$$\mathbb{E}_u\exp\big(\gamma\tau(R)\big)\le C\,\mathfrak{w}_m(u),\quad u\in H. \tag{5.27}$$

Proof. See Proposition 5.1 in [14] for a similar result in the discrete-time case. The proof of (5.27) follows the same arguments. The idea is to establish the inequality for the first hitting time of a ball in H and then to use the regularising property of the Navier–Stokes system.

Step 1: Hyper-exponential recurrence in H. For any $r > 0$, we denote by $\tau_0(r)$ the first hitting time of the ball $B_H(r)$:

$$\tau_0(r) = \inf\{t \geq 0 : u_t \in B_H(r)\}.$$

Let us prove that, for any $\gamma > 0$, there are positive numbers m, r, and C such that

$$\mathbb{E}_u \exp\bigl(\gamma\tau_0(r)\bigr) \leq C\,\mathfrak{w}_m(u), \quad u \in H. \tag{5.28}$$

Indeed, let $m > 0$ be so large that $q := 2e^{-m\alpha_1} < 1$. Then, by (5.23), we have

$$\mathbb{E}_u\|u_1\|^{2m} \leq q\left(\|u\|^{2m} \vee r\right), \quad u \in H, \tag{5.29}$$

where $r = e^{m\alpha_1}C_4(m, B_0, \|h\|)$. The Markov property and (5.29) imply that

$$p_k(u) := \mathbb{E}_u\bigl(I_{\{\tau_0(r)>k\}}\|u_k\|^{2m}\bigr) \leq q^k\|u\|^{2m}, \quad k \geq 0,\ u \in H$$

(cf. proof of Lemma 3.6.1 in [20]), hence

$$\mathbb{P}_u\{\tau_0(r) > k\} \leq r^{-2m}p_k(u) \leq r^{-2m}q^k\|u\|^{2m}. \tag{5.30}$$

As $q < 1$, the Borel–Cantelli lemma gives that $\mathbb{P}_u\{\tau_0(r) < \infty\} = 1$. Choosing $m \geq 1$ so large that $e^\gamma q < 1$, we derive from (5.30)

$$\begin{aligned}\mathbb{E}_u \exp\bigl(\gamma\tau_0(r)\bigr) &\leq 1 + \sum_{k=1}^{\infty} e^{\gamma k}\mathbb{P}_u\{\tau_0(r) > k-1\}\\ &\leq 1 + r^{-2m}\|u\|^{2m}\sum_{k=1}^{\infty} e^{\gamma k}q^{k-1} \leq C\,\mathfrak{w}_m(u),\end{aligned}$$

which proves (5.28).

Step 2: Hyper-exponential recurrence in H^1. First note that, for any numbers $p \in (0,1)$ and $r > 0$, there is $R > 0$ such that

$$\mathbb{P}_u\bigl\{u_1 \in X_R\bigr\} \geq 1 - p, \quad u \in B_H(r). \tag{5.31}$$

Indeed, this follows immediately from the Chebyshev inequality and (5.24):

$$\mathbb{P}_u\bigl\{\|u_1\|_1 > R\bigr\} \leq R^{-2}\mathbb{E}_u\|u_1\|_1^2 \leq CR^{-2}\|u\|^8 \leq CR^{-2}r^8 \leq p$$

for any $u \in B_H(r)$ and sufficiently large $R = R(r,p)$.

Now we combine (5.28) and (5.31) to prove (5.27). We introduce the sequences of stopping times

$$\tau_0' = \tau_0(r), \quad \tau_n' = \inf\{t \geq \tau_{n-1}' + 1 : u_t \in B_H(r)\}, \quad n \geq 1$$

and $\tau_n = \tau'_n + 1$. Let

$$\hat{n} = \min\{n \geq 0 : \tau_n \in X_R\}.$$

From (5.31) and the strong Markov property we have

$$\mathbb{P}_u\{\hat{n} > k\} \leq (1-p)^k, \quad k \geq 0,\, u \in H,$$

so $\hat{n}$ is almost surely finite. For any integers $k, M \geq 1$, we can write

$$\begin{aligned}\mathbb{P}_u\big\{\tau(R) \geq M\big\} &= \mathbb{P}_u\big\{\tau(R) \geq M, \tau_k < M\big\} + \mathbb{P}_u\big\{\tau(R) \geq M, \tau_k \geq M\big\} \\ &\leq \mathbb{P}_u\big\{\tau(R) > \tau_k\big\} + \mathbb{P}_u\big\{\tau_k \geq M\big\}.\end{aligned}$$

Since $\{\tau(R) > \tau_k\} \subset \{\hat{n} > k\}$, the first probability is estimated by $(1-p)^k$. The second one is estimated using (5.28) and the strong Markov property

$$\mathbb{P}_u\big\{\tau_k \geq M\big\} \leq C_1^k\, \mathfrak{w}_m(u) e^{-3\gamma M},$$

where $C_1 > 0$ does not depend on $k, M \geq 1$ and $u \in H$. Thus, we obtain

$$\mathbb{P}_u\big\{\tau(R) \geq M\big\} \leq (1-p)^k + C_1^k\, \mathfrak{w}_m(u) e^{-3\gamma M}.$$

To complete the proof, it remains to choose appropriately the parameters k and R. We take $k \sim \varepsilon M$, where $\varepsilon > 0$ is so small that $\varepsilon \log C_1 \leq \gamma$, and $R > 0$ so large that $\varepsilon \log(1-p)^{-1} \geq 2\gamma$. Then

$$\mathbb{P}_u\big\{\tau(R) \geq M\big\} \leq 2e^{-2\gamma M} \mathfrak{w}_m(u),$$

which implies (5.27).

□

5.5 Generalised Markov semigroups

For the reader's convenience, we recall here a result on the large-time asymptotics of generalised Markov semigroups in a Polish space X. It is established in [14] in the discrete-time setting, then extended to the continuous-time in [22]. Let us first recall some terminology.

Definition 5.5. We shall say that $\{P_t(u, \cdot), u \in X, t \geq 0\}$ is a *generalised Markov family of transition kernels* if the following two properties are satisfied.

Feller property. For any $t \geq 0$, the function $u \mapsto P_t(u, \cdot)$ is continuous from X to $\mathcal{M}_+(X)$ and does not vanish.

Kolmogorov–Chapman relation. For any $t, s \geq 0, u \in X$, and Borel set $\Gamma \subset X$, the following relation holds

$$P_{t+s}(u, \Gamma) = \int_X P_s(v, \Gamma) P_t(u, \mathrm{d}v).$$

To any such family we associate two semigroups by the following relations:

$$\mathfrak{P}_t : C_b(X) \to C_b(X), \qquad \mathfrak{P}_t\psi(u) = \int_X \psi(v)P_t(u, \mathrm{d}v),$$
$$\mathfrak{P}_t^* : \mathcal{M}_+(X) \to \mathcal{M}_+(X), \qquad \mathfrak{P}_t^*\mu(\Gamma) = \int_X P_t(v, \Gamma)\mu(\mathrm{d}v), \quad t \geq 0.$$

For a measurable function $\mathfrak{w} : X \to [1, +\infty]$ and a family $\mathcal{C} \subset C_b(X)$, we denote by $\mathcal{C}^{\mathfrak{w}}$ the set of functions $\psi \in L^\infty_{\mathfrak{w}}(X)$ that can be approximated with respect to the norm $\|\cdot\|_{L^\infty_{\mathfrak{w}}}$ by finite linear combinations of functions from $\mathcal{C}$. We shall say that a family $\mathcal{C} \subset C_b(X)$ is *determining* if for any $\mu, \nu \in \mathcal{M}_+(X)$ satisfying $\langle \psi, \mu \rangle = \langle \psi, \nu \rangle$ for all $\psi \in \mathcal{C}$, we have $\mu = \nu$. Finally, a family of functions $\psi_t : X \to \mathbb{R}$ is *uniformly equicontinuous* on a subset $K \subset X$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that $|\psi_t(u) - \psi_t(v)| < \varepsilon$ for any $u \in K$, $v \in B_X(u, \delta) \cap K$, and $t \geq 0$. The following result is Theorem 7.4 in [22].

Theorem 5.6. *Let $\{P_t(u, \cdot), u \in X, t \geq 0\}$ be a generalised Markov family of transition kernels satisfying the following four properties.*

Growth conditions. *There is an increasing sequence $\{X_R\}_{R=1}^\infty$ of compact subsets of X such that $X_\infty := \cup_{R=1}^\infty X_R$ is dense in X. The measures $P_t(u, \cdot)$ are concentrated on X_∞ for any $u \in X$ and $t > 0$, and there is a measurable function $\mathfrak{w} : X \to [1, +\infty]$ and an integer $R_0 \geq 1$ such that*[10]

$$\sup_{t \geq 0} \frac{\|\mathfrak{P}_t \mathfrak{w}\|_{L^\infty_{\mathfrak{w}}}}{\|\mathfrak{P}_t \mathbf{1}\|_{R_0}} < \infty,$$
$$\sup_{t \in [0,1]} \|\mathfrak{P}_t \mathbf{1}\|_\infty < \infty,$$

where $\|\cdot\|_R$ and $\|\cdot\|_\infty$ denote the L^∞ norm on X_R and X, respectively, and we set $\infty/\infty = 0$.

Time-continuity. *For any $g \in C_{\mathfrak{w}}(X)$ and $u \in X$, the function $t \mapsto \mathfrak{P}_t g(u)$ is continuous from $\mathbb{R}_+$ to $\mathbb{R}$.*

Uniform irreducibility. *For sufficiently large $\rho \geq 1$, any $R \geq 1$ and $r > 0$, there are positive numbers $l = l(\rho, r, R)$ and $p = p(\rho, r)$ such that*

$$P_l(u, B_X(\hat{u}, r)) \geq p \quad \text{for all } u \in X_R,\ \hat{u} \in X_\rho.$$

Uniform Feller property. *There is a number $R_0 \geq 1$ and a determining family $\mathcal{C} \subset C_b(X)$ such that $\mathbf{1} \in \mathcal{C}$ and the family $\{\|\mathfrak{P}_t \mathbf{1}\|_R^{-1} \mathfrak{P}_t \psi, t \geq 0\}$ is uniformly equicontinuous on X_R for any $\psi \in \mathcal{C}$ and $R \geq R_0$.*

Then for any $t > 0$, there is at most one measure $\mu_t \in \mathcal{P}_{\mathfrak{w}}(X)$ such that $\mu_t(X_\infty) = 1$ and

$$\mathfrak{P}_t^* \mu_t = \lambda(t)\mu_t \quad \text{for some } \lambda(t) \in \mathbb{R}$$

[10] The expression $(\mathfrak{P}_t \mathfrak{w})(u)$ is understood as an integral of a positive function $\mathfrak{w}$ against a positive measure $P_t(u, \cdot)$.

satisfying the following condition:

$$\|\mathfrak{P}_t\mathfrak{w}\|_R \int_{X\setminus X_R} \mathfrak{w}(u)\,\mu_t(\mathrm{d}u) \to 0 \quad \textit{as } R\to\infty.$$

Moreover, if such a measure μ_t exists for all $t>0$, then it is independent of t (we set $\mu := \mu_t$), the corresponding eigenvalue is of the form $\lambda(t)=\lambda^t$, $\lambda>0$, $\operatorname{supp}\mu = X$*, there is a non-negative function $h\in L^\infty_{\mathfrak{w}}(X)$ such that $\langle h,\mu\rangle = 1$,*

$$(\mathfrak{P}_t h)(u) = \lambda^t h(u) \quad \textit{for } u\in X,\ t>0,$$

the restriction of h to X_R belongs to $C_+(X_R)$, and for any $\psi\in\mathcal{C}^{\mathfrak{w}}$ and $R\geq 1$, we have

$$\lambda^{-t}\mathfrak{P}_t\psi \to \langle\psi,\mu\rangle h \quad \textit{in } C(X_R)\cap L^1(X,\mu) \textit{ as } t\to\infty. \tag{5.32}$$

Finally, if a Borel set $B\subset X$ is such that

$$\sup_{u\in B}\int_{X\setminus X_R}\mathfrak{w}(v)\,P_s(u,\mathrm{d}v)\to 0 \quad \textit{as } R\to\infty \tag{5.33}$$

for some $s>0$, then

$$\lambda^{-t}\mathfrak{P}_t\psi \to \langle\psi,\mu\rangle h \quad \textit{in } L^\infty(B) \textit{ as } t\to\infty$$

for any $\psi\in\mathcal{C}^{\mathfrak{w}}$.

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